Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Technical Report No. 498

Eliminating Implicit Information Leaks by Transformational Typing and Unification

Boris Köpf[1] and Heiko Mantel[2]*

[1] Information Security, ETH Zürich, Switzerland, Boris.Koepf@inf.ethz.ch
[2] Department of Computer Science, RWTH Aachen University, Germany, mantel@cs.rwth-aachen.de

Abstract. Before starting the security analysis of an existing system, the most likely outcome is often already clear, namely that the system is not entirely secure. Modifying a program such that it passes the analysis is a difficult problem and usually left entirely to the programmer. In this article, we show that and how unification can be used to compute such program transformations. This opens a new perspective on the problem of correcting insecure programs. We demonstrate that integrating our approach into an existing transforming type system can also improve the precision of the analysis and the quality of the resulting programs.

1 Introduction

Security requirements like confidentiality or integrity can often be adequately expressed by restrictions on the permitted flow of information. This approach goes beyond access control models in that it controls not only the access to data, but also how data is propagated within a program after a legitimate access.

Security type systems provide a basis for automating the information flow analysis of concrete programs [SM03]. If type checking succeeds then a program has secure information flow. If type checking fails then the program might be insecure and should not be run. After a failed type check, the task of correcting the program is often left to the programmer. Given the significance of the problem, it would be very desirable to have automated tools that better support the programmer in this task. For the future, we envision a framework for the information flow analysis that, firstly, gives more constructive advice on how a given program could be improved and, secondly, in some cases automatically corrects the program, or parts thereof, without any need for interaction by the programmer. The current article focuses on the second of these two aspects.

* Most of this work has been performed while being a member of the Information Security Group at the ETH Zurich. The author gratefully acknowledges support by the DFG.

Obviously, one cannot allow an automatic transformation to modify programs in completely arbitrary ways as the transformed program should resemble the original program in some well-defined way. Such constraints can be captured by defining an equivalence relation on programs and demanding that the transformed program is equivalent to the original program under this relation. A second equivalence relation can be used to capture the objective of a transformation. The problem of removing implicit information leaks from a program can be viewed as the problem of making alternative execution paths observationally equivalent. For instance, if the guard of a conditional depends on a secret then the two branches must be observationally equivalent because, otherwise, an untrusted observer might be able to deduce the value of the guard and, thereby, the secret. The PER model [SS99] even reduces the problem of making an entire program secure to the problem of making the program equivalent to itself.

In our approach, meta-variables are inserted into a program and are instantiated with programs during the transformation. The problem of making two program fragments equivalent is cast as a unification problem, which allows us to automatically compute suitable substitutions using existing unification algorithms. The approach is parametric in two equivalence relations. The first relation captures the semantic equivalence to be preserved by the transformation while the second relation captures the observational equivalence to be achieved.

We define two concrete equivalence relations to instantiate our approach and integrate this instance into an existing transforming type system [SS00]. This results in a security type system that is capable of recognizing some secure programs and of correcting some insecure programs that are rejected by the original type system. Moreover, the resulting programs are faster and often substantially smaller in size. Another advantage over the cross-copying technique [Aga00], which constitutes the current state of the art in this area, is that security policies with more than two levels can be considered. Besides these technical advantages, the use of unification yields a very natural perspective on the problem of making two programs observationally equivalent. However, we do not claim that using unification will solve all problems with repairing insecure programs or that unification would be the only way to achieve the above technical advantages.

The contributions of this article are a novel approach to making the information flow in a given program secure and the demonstration that transforming security type systems can benefit from the integration of this approach.

2 The Approach

The observational capabilities of an attacker can be captured by an equivalence relation on configurations, i.e. pairs consisting of a program and a state. Namely, (C_1, s_1) is observationally equivalent to (C_2, s_2) for an attacker a if and only if the observations that a makes when C_1 is run in state s_1 equal a's observations when C_2 is run in s_2. The programs C_1 and C_2 are observationally equivalent for a if, for all states s_1 and s_2 that are indistinguishable for a, the configurations (C_1, s_1) and (C_2, s_2) are observationally equivalent for a. The resulting relation on programs is only a partial equivalence relation (PER), i.e. a transitive and symmetric relation that need not be reflexive. If a program C is not observationally equivalent to itself for a then running C in two indistinguishable states may lead to different observations and, thereby, reveal the differences between the states or, in other words, let a learn secret information. This observation is the key to capturing secure information flow in the PER model [SS99] in which a program is secure if and only if it is observationally equivalent to itself.

In this article, we focus on the removal of implicit information leaks from a program. There is a danger of implicit information leakage if the flow of control depends on a secret and the alternative execution paths are not observationally equivalent for an attacker. The program $\mathsf{if}\ h\ \mathsf{then}\ l{:=}1\ \mathsf{else}\ l{:=}0$, for instance, causes information to flow from the boolean guard h into the variable l, and this constitutes an illegitimate information leak if h stores a secret and the value of l is observable for the attacker. Information can also be leaked in a similar way, e.g., when the guard of a loop depends on a secret, when it depends on a secret whether an exception is raised, or when the target location of a jump depends on a secret. For brevity of the presentation, we focus on the case of conditionals.

We view the problem of making the branches of a conditional equivalent as a unification problem under a theory that captures observational equivalence. To this end, we insert meta-variables into the program under consideration that can be substituted during the transformation. For a given non-transforming security type system, the rule for conditionals is modified such that, instead of checking whether the branches are equivalent, the rule calculates a unifier of the branches and applies it to the conditional. Typing rules for other language constructs are lifted such that they propagate the transformations that have occurred in the analysis of the subprograms. In summary, our approach proceeds as follows:

1. Lift the given program by inserting meta-variables at suitable locations.
2. Repair the lifted program by applying lifted typing rules.
3. Eliminate all remaining meta-variables.

The approach is not only parametric in the given security type system and in the theory under which branches are unified, but also in where meta-variables are placed and how they may be substituted. The latter two parameters determine how similar a transformed program is to the original program. They also limit the extent to which insecure programs can be corrected. For instance, one might decide to insert meta-variables between every two sub-commands and to permit the substitution of meta-variables with arbitrary programs. For these choices, lifting $P_1 = \mathsf{if}\ h\ \mathsf{then}\ l{:=}1\ \mathsf{else}\ l{:=}0$ results in $\mathsf{if}\ h\ \mathsf{then}\ (\alpha_1; l{:=}1; \alpha_2)\ \mathsf{else}\ (\alpha_3; l{:=}0; \alpha_4)$ and the substitution $\{\alpha_1 \backslash l{:=}0,\ \ \alpha_2 \backslash \epsilon, \alpha_3 \backslash \epsilon,\ \alpha_4 \backslash l{:=}1\}$ (where ϵ is denotes the empty program) is a unifier of the branches under any equational theory as the substituted program is $\mathsf{if}\ h\ \mathsf{then}\ (l{:=}0; l{:=}1)\ \mathsf{else}\ (l{:=}0; l{:=}1)$. Alternatively, one might decide to restrict the range of substitutions to sequences of skip statements. This ensures that the transformed program more closely resembles the original program, essentially any transformed program is a slowed-down version of the original program, but makes it impossible to correct programs like P_1. However, the program $P_2 = \mathsf{if}\ h\ \mathsf{then}\ (\mathsf{skip}; l{:=}1)\ \mathsf{else}\ l{:=}1$, which is insecure in a multi-threaded setting (as we will explain later in this section), can be corrected under these choices to if h then (skip; l:=1) else (skip; l:=1). Alternatively, one could even decide to insert higher-order meta-variables such that lifting P_1 leads to if h then $\alpha_1(l{:=}1)$ else $\alpha_2(l{:=}0)$ and applying, e.g., the unifier $\{\alpha_1\backslash(\lambda x.\mathsf{skip}),\ \alpha_2\backslash(\lambda x.\mathsf{skip})\}$ results in if h then skip else skip while applying the unifier $\{\alpha_1\backslash(\lambda x.x),\ \alpha_2\backslash(\lambda x.l{:=}1)\}$ results in if h then l:=1 else l:=1. These examples just illustrate the wide spectrum of possible choices for defining in which sense a transformed program must be equivalent to the original program. Ultimately it depends on the application, how flexible one is in dealing with the trade-off between being able to correct more insecure programs and having transformed programs that more closely resemble the original programs.

There also is a wide spectrum of possible choices for defining the (partial) observational equivalence relation. For simplicity, assume that variables are classified as either low or high depending on whether their values are observable by the attacker (low variables) or secret (high variables). As a convention, we denote low variables by l and high variables by h, possibly with indexes and primes. Given that the values of low variables are only observable at the end of a program run, the programs $P_3 = (\mathsf{skip}; l := 0)$ and $P_4 = (l := h; l := 0)$ are observationally equivalent and each is equivalent to itself (which means secure information flow in the PER model). However, if the attacker can observe also the intermediate values of low variables then they are not equivalent and, moreover, only P_3 is secure while P_4 is insecure. If the attacker can observe the timing of assignments or the duration of a program run then P_2 = if h then (skip; l:=1) else l:=1 is insecure and, hence, not observationally equivalent to itself. In a multi-threaded setting, P_2 should be considered insecure even if the attacker cannot observe the timing of assignments or the duration of a program run. If $P_3 = (\mathsf{skip}; l := 0)$ is run in parallel with P_2 under a shared memory and a round-robin scheduler that re-schedules after every sub-command then the final value of l is 0 and 1 if the initial value of h is 0 and 1, respectively. That is, a program that is observationally equivalent to itself in a sequential setting might not be observationally equivalent to itself in a multi-threaded setting – for the same attacker.

3 Instantiating the Approach

We are now ready to illustrate how our approach can be instantiated. We introduce a simple programming language, a security policy, an observational equivalence, and a program equivalence to be preserved under the transformation.

Programming Language. We adopt the multi-threaded while language (short: MWL) from [SS00], which includes assignments, conditionals, loops, and a command for dynamic thread creation. The set *Com* of commands is defined by

$$C ::= \mathsf{skip} \mid Id{:=}Exp \mid C_1; C_2 \mid \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \mid \mathsf{while}\ B\ \mathsf{do}\ C \mid \mathsf{fork}(CV)$$

where V is a command vector in $\boldsymbol{Com} = \bigcup_{n\in\mathbb{N}} Com^n$. *Expressions* are variables, constants, or terms resulting from applying binary operators to expressions. A *state* is a mapping from variables in a given set *Var* to values in a given set *Val*. We use the judgment $\langle Exp, s\rangle \downarrow n$ for specifying that expression *Exp* evaluates to value n in state s. Expression evaluation is assumed to be total and to occur atomically. We say that expressions *Exp* and Exp' are *equivalent* to each other (denoted by $Exp \equiv Exp'$) if and only if they evaluate to identical values in each state, i.e. $\forall s \in S : \forall v \in Val : \langle\!| Exp, s |\!\rangle \downarrow v \Leftrightarrow \langle\!| Exp', s |\!\rangle \downarrow v$.

The operational semantics for MWL is formalized in Figures 5 and 6 in the appendix. *Deterministic judgments* have the form $\langle\!| C, s |\!\rangle \twoheadrightarrow \langle\!| W, t |\!\rangle$ expressing that command C performs a computation step in state s, yielding a state t and a vector of commands W, which has length zero if C terminated, length one if it has neither terminated nor spawned any threads, and length > 1 if threads were spawned. That is, a command vector of length n can be viewed as a *pool of n threads* that run concurrently. *Nondeterministic judgments* have the form $\langle\!| V, s |\!\rangle \rightarrow \langle\!| V', t |\!\rangle$ expressing that some thread C_i in the thread pool V performs a step in state s resulting in the state t and some thread pool W. The global thread pool V' results then by replacing C_i with W. For simplicity, we do not distinguish between commands and command vectors of length one in the notation and use the term *program* for referring to commands as well as to command vectors. A *configuration* is then a pair $\langle\!| V, s |\!\rangle$ where V specifies the threads that are currently active and s defines the current state of the memory.

In the following, we adopt the naming conventions used above. That is, s, t denote states, *Exp* denotes an expression, B denotes a boolean expression, C denotes a command, and V, W denote command vectors.

Security Policy and Labellings. We assume a two-domain security policy, where the requirement is that there is no flow of information from the *high domain* to the *low domain.* This is the simplest policy under which the problem of secure information flow can be studied. Each program variable is associated with a security domain by means of a *labeling* $lab : Var \rightarrow \{low, high\}$. The intuition is that values of *low* variables can be observed by the attacker and, hence, should only be used to store public data. *High variables* are used for storing secret data and, hence, their values must not be observable for the attacker. As mentioned before, we use l and h to denote high and low variables, respectively. An expression *Exp* has the security domain *low* (denoted by $Exp : low$) if all variables in *Exp* have domain *low* and, otherwise, has security domain *high* (denoted by $Exp : high$). The intuition is that values of expressions with domain *high* possibly depend on secrets while values of *low* expressions can only depend on public data.

Observational Equivalence. The rules in Figure 1 inductively define a relation $\doteq_L \subseteq \boldsymbol{Com} \times \boldsymbol{Com}$ that will serve us as an observational equivalence relation.

The relation $\doteq_L$ captures observational equivalence for an attacker who can see the values of low variables at any point during a program run and cannot distinguish states s_1 and s_2 if they are low equal (denoted by $s_1 =_L s_2$), i.e. if $\forall var \in Var : lab(var) = low \implies s_1(var) = s_2(var)$. He cannot distinguish two program runs that have equal length and in which every two corresponding states are low equal. For capturing this intuition, Sabelfeld and Sands introduce the notion of a strong low bisimulation. The relation $\doteq_L$ also captures this intuition and, moreover, programs that are related by $\doteq_L$ are also strongly bisimilar. That is, $\doteq_L$ is a decidable approximation of the strong bisimulation relation.

Definition 1 ([SS00]). *The* strong low-bisimulation $\cong_L$ *is the union of all symmetric relations R on command vectors $V, V' \in$ **Com** of equal size, i.e. $V = \langle C_1, \ldots, C_n \rangle$ and $V' = \langle C'_1, \ldots, C'_n \rangle$, such that*

$$\begin{array}{l}\forall s, s', t \in S : \forall i \in \{1 \ldots n\} : \forall W \in \textbf{\textit{Com}}: \\ \ [(V \, R \, V' \wedge s =_L s' \wedge \langle\!| C_i, s |\!\rangle \twoheadrightarrow \langle\!| W, t |\!\rangle) \\ \quad \Rightarrow \exists W' \in \textbf{\textit{Com}}: \exists t' \in S: (\langle\!| C'_i, s' |\!\rangle \twoheadrightarrow \langle\!| W', t' |\!\rangle \wedge W \, R \, W' \wedge t =_L t')]\end{array}$$

Theorem 1 (Adequacy of $\simeq_L$). *If $V \simeq_L V'$ is derivable then $V \cong_L V'$ holds.*

The proofs of this and all subsequent results will be provided in an extended version of this article.

Remark 1. Note that $\simeq_L$ and $\cong_L$ are only partial equivalence relations, i.e. they are transitive and symmetric, but not reflexive. For instance, the program $l{:=}h$ is not $\simeq_L$-related to itself because the precondition of $[LA]$, the only rule in Figure 1 applicable to assignments to low variables, rules out that high variables occur on the right hand side of the assignment. Moreover, the program $l{:=}h$ is not strongly low bisimilar to itself because the states s and t (defined by $s(l) = 0$, $s(h) = 0$, $t(l) = 0$, $t(h) = 1$) are low equal, but the states s' and t' resulting after $l{:=}h$ is run in s and t, respectively, are not low equal ($s'(l) = 0 \neq 1 = t'(l)$).

However, $\simeq_L$ is an equivalence relation if one restricts programs to the language *Slice* that we define as the largest sub-language of *Com* without assignments of high expressions to low variables, assignments to high variables, and loops or conditionals having high guards. On *Slice*, $\simeq_L$ even constitutes a congruence relation. This sub-language is the context in which we will apply unification and, hence, using the term *unification under an equational theory* is justified. $\Diamond$

Program Equivalence. We introduce an equivalence relation $\simeq$ to constrain the modifications caused by the transformation. Intuitively, this relation requires a transformed program to be a slowed down version of the original program. This is stronger than the constraint in [SS00].

Definition 2. *The* weak possibilistic bisimulation $\simeq$ *is the union of all symmetric relations R on command vectors such that whenever $V \; R \; V'$ then for all states s, t and all vectors W there is a vector W' such that*

$$\begin{array}{l}\langle\!| V, s |\!\rangle \twoheadrightarrow \langle\!| W, t |\!\rangle \implies (\langle\!| V', s |\!\rangle \twoheadrightarrow^* \langle\!| W', t |\!\rangle \wedge W R W') \\ \textit{and } V = \langle\rangle \implies \langle\!| V', s |\!\rangle \twoheadrightarrow^* \langle\!| \langle\rangle, s |\!\rangle \ .\end{array}$$

4 Lifting a Security Type System

In this section we introduce a formal framework for transforming programs by inserting and instantiating meta-variables. Rather than developing an entirely new formalism from scratch, we adapt an existing security type system from [SS00]. We show that any transformation within our framework is sound in the sense that the output is secure and the behavior of the original program is preserved in the sense of Definition 2.

$$\frac{}{\mathsf{skip} \simeq_L \mathsf{skip}}\ [\mathit{Skip}] \qquad \frac{\mathit{Id} : \mathit{high}}{\mathsf{skip} \simeq_L \mathit{Id}{:=}\mathit{Exp}}\ [\mathit{SHA_1}] \qquad \frac{\mathit{Id} : \mathit{high}}{\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathsf{skip}}\ [\mathit{SHA_2}]$$

$$\frac{\mathit{Id} : \mathit{high} \quad \mathit{Id}' : \mathit{high}}{\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathit{Id}'{:=}\mathit{Exp}'}\ [\mathit{HA}] \qquad \frac{\mathit{Id} : \mathit{low} \quad \mathit{Exp} : \mathit{low} \quad \mathit{Exp}' : \mathit{low} \quad \mathit{Exp} \equiv \mathit{Exp}'}{\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathit{Id}{:=}\mathit{Exp}'}\ [\mathit{LA}]$$

$$\frac{C_1 \simeq_L C_1', \ldots, C_n \simeq_L C_n'}{\langle C_1, \ldots, C_n \rangle \simeq_L \langle C_1', \ldots, C_n' \rangle}\ [\mathit{PComp}] \qquad \frac{C \simeq_L C' \quad V \simeq_L V'}{\mathsf{fork}(CV) \simeq_L \mathsf{fork}(C'V')}\ [\mathit{Fork}]$$

$$\frac{B, B' : \mathit{low} \quad B \equiv B' \quad C_1 \simeq_L C_1' \quad C_2 \simeq_L C_2'}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \simeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'}\ [\mathit{LIte}] \qquad \frac{B, B' : \mathit{low} \quad B \equiv B' \quad C \simeq_L C'}{\mathsf{while}\ B\ \mathsf{do}\ C \simeq_L \mathsf{while}\ B'\ \mathsf{do}\ C'}\ [\mathit{WL}]$$

$$\frac{B, B' : \mathit{high} \quad C_1 \simeq_L C_1' \quad C_1 \simeq_L C_2' \quad C_1 \simeq_L C_2}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \simeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'}\ [\mathit{HIte}] \qquad \frac{C_1 \simeq_L C_1' \quad C_2 \simeq_L C_2'}{C_1; C_2 \simeq_L C_1'; C_2'}\ [\mathit{SComp}]$$

$$\frac{B' : \mathit{high} \quad C_1 \simeq_L C_1' \quad C_1 \simeq_L C_2'}{\mathsf{skip}; C_1 \simeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'}\ [\mathit{SHIte_1}] \qquad \frac{B : \mathit{high} \quad C_1 \simeq_L C_1' \quad C_2 \simeq_L C_1'}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \simeq_L \mathsf{skip}; C_1'}\ [\mathit{SHIte_2}]$$

$$\frac{\mathit{Id} : \mathit{high} \quad B' : \mathit{high} \quad C_1 \simeq_L C_1' \quad C_1 \simeq_L C_2'}{\mathit{Id}{:=}\mathit{Exp}; C_1 \simeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'}\ [\mathit{HAHIte_1}]$$

$$\frac{\mathit{Id}' : \mathit{high} \quad B : \mathit{high} \quad C_1 \simeq_L C_1' \quad C_2 \simeq_L C_1'}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \simeq_L \mathit{Id}'{:=}\mathit{Exp}'; C_1'}\ [\mathit{HAHIte_2}]$$

Fig. 1. A notion of observational equivalence

Substitutions and Liftings. We insert meta-variables from a set $\mathcal{V} = \{\alpha_1, \alpha_2, \dots\}$ into a program by sequential composition with its sub-terms. The extension of MWL with meta-variables is denoted by $\mathrm{MWL}_\mathcal{V}$. The set $\mathit{Com}_\mathcal{V}$ of commands in $\mathrm{MWL}_\mathcal{V}$ is defined by[3]

$$\begin{aligned} C ::= {} & \mathsf{skip} \mid \mathit{Id}{:=}\mathit{Exp} \mid C_1; C_2 \mid C; X \mid X; C \\ & \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \mid \mathsf{while}\ B\ \mathsf{do}\ C \mid \mathsf{fork}(CV)\ , \end{aligned}$$

where placeholders X, Y range over $\mathcal{V}$. Analogously to MWL, the set of all command vectors in $\mathrm{MWL}_\mathcal{V}$ is defined by $\boldsymbol{Com_\mathcal{V}} = \bigcup_{n \in \mathbb{N}} (\mathit{Com}_\mathcal{V})^n$. Note that the *ground programs* in $\mathrm{MWL}_\mathcal{V}$ are exactly the programs in MWL. The operational semantics for such programs remain unchanged, whereas programs with meta-variables are not meant to be executed.

Meta-variables may be substituted with programs, meta-variables or the special symbol ϵ that acts as the neutral element of the sequential composition operator (";"), i.e. $\epsilon; C = C$ and $C; \epsilon = C$[4]. When talking about programs in $\mathit{Com}_\mathcal{V}$ under a given substitution, we implicitly assume that these equations have been applied (from left to right) to eliminate the symbol ϵ from the program. Moreover, we view sequential composition as an associative operator and implicitly identify programs that differ only in the use of parentheses for sequential composition. That is, $C_1; (C_2; C_3)$ and $(C_1; C_2); C_3$ denote the same program.

[3] Here and in the following, we overload notation by using C and V to denote commands and command vectors in $\mathit{Com}_\mathcal{V}$, respectively.

[4] Note that skip is not a neutral element of (";") as skip requires a computation step.

A mapping $\sigma : \mathcal{V} \rightarrow (\{\epsilon\} \cup \mathcal{V} \cup Com_{\mathcal{V}})$ is a *substitution* if the set $\{\alpha \in \mathcal{V} \mid \sigma(\alpha) \neq \alpha\}$ is finite. A substitution mapping each meta-variable in a program V to $\{\epsilon\} \cup Com$ is a *ground substitution of* V. A substitution π mapping all meta-variables in V to ϵ is a *projection of* V. Given a program V in $\boldsymbol{Com}$, we call every program V' in $\boldsymbol{Com_{\mathcal{V}}}$ with $\pi V' = V$ a *lifting of* V.

For example, the program if h then $(\alpha_1; \mathsf{skip}; \alpha_2; l{:=}1)$ else $(\alpha_3; l{:=}1)$ is in fact a lifting of if h then $(\mathsf{skip}; l{:=}1)$ else $l{:=}1$. In the remainder of this article, we will focus on substitutions with a restricted range.

Definition 3. *A substitution with range* $\{\epsilon\} \cup Stut_{\mathcal{V}}$ *is called* preserving*, where* $Stut_{\mathcal{V}}$ *is defined by* $C ::= X \mid \mathsf{skip} \mid C_1; C_2$ *(the* C_i *range over* $Stut_{\mathcal{V}}$*).*

The term *preserving* substitution is justified by the fact that such substitutions preserve a given program's semantics as specified in Definition 2.

Theorem 2 (Preservation of Behavior).

1. *Let* $V \in \boldsymbol{Com_{\mathcal{V}}}$*. For all preserving substitutions* σ, ρ *that are ground for* V*, we have* $\sigma(V) \simeq \rho(V)$*.*
2. *Let* $V \in \boldsymbol{Com}$*. For each lifting* V' *of* V *and each preserving substitution* σ *with* $\sigma(V')$ *ground, we have* $\sigma(V') \simeq V$*.*

Unification of Programs. The problem of finding a substitution that makes the branches of conditionals with high guards observationally equivalent can be viewed as the problem of finding a unifier for the branches under the equational theory $\simeq_L$.[5] To this end, we lift the relation $\simeq_L \subseteq \boldsymbol{Com} \times \boldsymbol{Com}$ to a binary relation on $\boldsymbol{Com_{\mathcal{V}}}$ that we also denote by $\simeq_L$.

Definition 4. $V_1, V_2 \in \boldsymbol{Com_{\mathcal{V}}}$ *are* observationally equivalent *(*$V_1 \simeq_L V_2$*) iff* $\sigma V_1 \simeq_L \sigma V_2$ *for each preserving substitution* σ *that is ground for* V_1 *and* V_2*.*

Definition 5. *A* $\simeq_L$-unification problem Δ *is a finite set of statements of the form* $V_i \simeq^?_L V'_i$*, i.e.* $\Delta = \{V_0 \simeq^?_L V'_0, \ldots, V_n \simeq^?_L V'_n\}$ *with* $V_i, V'_i \in \boldsymbol{Com_{\mathcal{V}}}$ *for all* $i \in \{0, \ldots, n\}$*. A substitution* σ *is a* preserving unifier for Δ *if and only if* σ *is preserving and* $\sigma V_i \simeq_L \sigma V'_i$ *holds for each* $i \in \{0, \ldots, n\}$*. A* $\simeq_L$*-unification problem is* solvable *if the set of preserving unifiers* $\mathcal{U}(\Delta)$ *for* Δ *is not empty.*

A Transforming Type System. The transforming type system in Figure 2 has been derived from the one in [SS00]. We use the judgment $V \hookrightarrow V' : S$ for denoting that the MWL$_{\mathcal{V}}$-program V can be transformed into an MWL$_{\mathcal{V}}$-program V'. The intention is that V' has secure information flow and reflects the semantics of V as specified by Definition 2. The *slice* S is a program that is in the sub-language $Slice_{\mathcal{V}}$ and describes the timing behavior of V'. The novelty over [SS00] is that our type system operates on $\boldsymbol{Com_{\mathcal{V}}}$ (rather than on $\boldsymbol{Com}$) and that the rule for high conditionals has been altered. In the original type system, a high conditional is transformed by sequentially composing each branch with the

[5] The term *equational theory* is justified as we apply unification only to programs in the sub-language $Slice_{\mathcal{V}}$ for which $\simeq_L$ constitutes a congruence relation (see Remark 1).

$$\frac{}{\mathsf{skip} \hookrightarrow \mathsf{skip} : \mathsf{skip}}\ [Skp] \qquad \frac{Id : high}{Id{:=}Exp \hookrightarrow Id{:=}Exp : \mathsf{skip}}\ [Ass_h] \qquad \frac{C_1 \hookrightarrow C_1' : S_1 \quad C_2 \hookrightarrow C_2' : S_2}{C_1; C_2 \hookrightarrow C_1'; C_2' : S_1; S_2}\ [Seq]$$

$$\frac{Id : low \qquad Exp : low}{Id{:=}Exp \hookrightarrow Id{:=}Exp : Id{:=}Exp}\ [Ass_l] \qquad \frac{B : low \qquad C \hookrightarrow C' : S}{\mathsf{while}\ B\ \mathsf{do}\ C \hookrightarrow \mathsf{while}\ B\ \mathsf{do}\ C' : \mathsf{while}\ B\ \mathsf{do}\ S}\ [Whl]$$

$$\frac{C_1 \hookrightarrow C_1' : S_1 \quad \ldots \quad C_n \hookrightarrow C_n' : S_n}{\langle C_1, \ldots, C_n \rangle \hookrightarrow \langle C_1', \ldots, C_n' \rangle : \langle S_1, \ldots, S_n \rangle}\ [Par] \qquad \frac{C_1 \hookrightarrow C_1' : S_1 \qquad V_2 \hookrightarrow V_2' : S_2}{\mathsf{fork}(C_1 V_2) \hookrightarrow \mathsf{fork}(C_1' V_2') : \mathsf{fork}(S_1 S_2)}\ [Frk]$$

$$\frac{B : low \qquad C_1 \hookrightarrow C_1' : S_1 \qquad C_2 \hookrightarrow C_2' : S_2}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \hookrightarrow \mathsf{if}\ B\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2' : \mathsf{if}\ B\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2}\ [Cond_l]$$

$$\frac{B : high \quad C_1 \hookrightarrow C_1' : S_1 \quad C_2 \hookrightarrow C_2' : S_2 \quad \sigma \in \mathcal{U}(\{S_1 \stackrel{?}{\simeq}_L S_2\})}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \hookrightarrow \mathsf{if}\ B\ \mathsf{then}\ \sigma C_1'\ \mathsf{else}\ \sigma C_2' : \mathsf{skip}; \sigma S_1}\ [Cond_h] \qquad \frac{}{X \hookrightarrow X : X}\ [Var]$$

Fig. 2. A transforming security type system for programs with meta-variables

slice of the respective other branch. Instead of cross-copying slices, our rule instantiates the meta-variables occurring in the branches using preserving unifiers. The advantages of this modification are discussed in Section 6. Note that the rule $[Cond_h]$ does not mandate the choice of a specific preserving unifier of the branches. Nevertheless, we can prove that the type system meets our previously described intuition about the judgment $V \hookrightarrow V' : S$. To this end, we employ Sabelfeld and Sands's strong security condition for defining what it means for a program to have secure information flow. Many other definitions are possible (see e.g. [SM03]).

Definition 6. *A program $V \in \boldsymbol{Com}$* is strongly secure *if and only if $V \cong_L V$ holds. A program $V \in \boldsymbol{Com}_{\mathcal{V}}$* is strongly secure *if and only if σV is strongly secure for each substitution σ that is preserving and ground for V.*

Theorem 3 (Soundness Type System). *If $V \hookrightarrow V' : S$ can be derived then (1) V' has secure information flow, (2) $V \simeq V'$ holds,*[6] *and (3) $V' \cong_L S$ holds.*

The following corollary is an immediate consequence of Theorems 2 and 3. It shows that lifting a program and then applying the transforming type system preserves a program's behavior in the desired way.

Corollary 1. *If $V^* \hookrightarrow V' : S$ is derivable for some lifting $V^* \in \boldsymbol{Com}_{\mathcal{V}}$ of a program $V \in \boldsymbol{Com}$ then V' has secure information flow and $V \simeq V'$.*

5 Automating the Transformation

In Section 4, we have shown our type system to be sound for any choice of liftings and preserving unifiers in the applications of rule $[Cond_h]$. For automating the transformation, we have to define more concretely where meta-variables are inserted and how unifiers are determined.

[6] Here and in the following, we define $\simeq$ on $Com_{\mathcal{V}}$ by $C \simeq C'$ iff $\sigma C \simeq \sigma C'$ for any substitution σ that is preserving and ground for C and for C'.

$$\frac{X \text{ fresh}}{\mathsf{skip} \rightharpoonup \mathsf{skip}; X} \quad \frac{\mathit{Id} : \mathit{high} \quad X \text{ fresh}}{\mathit{Id}{:=}\mathit{Exp} \rightharpoonup \mathit{Id}{:=}\mathit{Exp}; X} \quad \frac{\mathit{Id} : \mathit{low} \quad X, Y \text{ fresh}}{\mathit{Id}{:=}\mathit{Exp} \rightharpoonup X; \mathit{Id}{:=}\mathit{Exp}; Y}$$

$$\frac{C_1 \rightharpoonup C_1'; X \quad C_2 \rightharpoonup C_2'}{C_1; C_2 \rightharpoonup C_1'; C_2'} \quad \frac{C_1 \rightharpoonup C_1', \ldots, \quad C_n \rightharpoonup C_n'}{\langle C_1, \ldots, C_n \rangle \rightharpoonup \langle C_1', \ldots, C_n' \rangle} \quad \frac{C_1 \rightharpoonup C_1' \quad V_2 \rightharpoonup V_2' \quad X, Y \text{ fresh}}{\mathsf{fork}(C_1 V_2) \rightharpoonup X; (\mathsf{fork}(C_1' V_2')); Y}$$

$$\frac{C \rightharpoonup C' \quad X, Y \text{ fresh}}{\mathsf{while}\ B\ \mathsf{do}\ C \rightharpoonup X; (\mathsf{while}\ B\ \mathsf{do}\ C'); Y} \quad \frac{C_1 \rightharpoonup C_1' \quad C_2 \rightharpoonup C_2' \quad X, Y \text{ fresh}}{\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \rightharpoonup X; (\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2); Y}$$

Fig. 3. A calculus for computing most general liftings

Automatic Insertion of Meta-Variables. When lifting a program, one is faced with a trade off: inserting meta-variables means to create possibilities for correcting the program, but it also increases the complexity of the unification problem. Within this spectrum our objective is to minimize the number of inserted meta-variables without losing the possibility of correcting the program.

To this end, observe that two programs C_1 and C_2 within the sub-language $\mathit{Pad}_{\mathcal{V}}$, the extension of $\mathit{Stut}_{\mathcal{V}}$ with high assignments, are related via $\doteq_L$ whenever they contain the same number of constants, i.e., skips and assignments to high variables (denoted as $\mathit{const}(C_1) = \mathit{const}(C_2)$), and the same number of occurrences of each meta-variable α (denoted by $|C_1|_\alpha = |C_2|_\alpha$). Note that the positioning of meta-variables is irrelevant.

Lemma 1. *For two commands C_1 and C_2 in $\mathit{Pad}_{\mathcal{V}}$ we have $C_1 \doteq_L C_2$ if and only if $\mathit{const}(C_1) = \mathit{const}(C_2)$ and $\forall \alpha \in \mathcal{V}\colon |C_1|_\alpha = |C_2|_\alpha$.*

Moreover, observe that inserting one meta-variable next to another does not create new possibilities for correcting a program. This, together with Lemma 1, implies that inserting one meta-variable into every subprogram within $\mathit{Pad}_{\mathcal{V}}$ is sufficient for allowing every possible correction. We use this insight to define a mapping $\rightharpoonup$: $\boldsymbol{Com} \to \boldsymbol{Com}_{\boldsymbol{\mathcal{V}}}$ that calculates a lifting of a program by inserting one fresh meta-variable at the end of every sub-program in $\mathit{Pad}_{\mathcal{V}}$, and between every two sub-programs outside $\mathit{Pad}_{\mathcal{V}}$. The mapping is defined inductively: A fresh meta-variable is sequentially composed to the right hand side of each subprogram. Another fresh meta-variable is sequentially composed to the left hand side of each assignment to a low variable, fork, while loop, or conditional. A lifting of a sequentially composed program is computed by sequentially composing the liftings of the subprograms while removing the terminal variable of the left program.The full calculus is given in Figure 3. The liftings computed by $\rightharpoonup$ are *most general* in the sense that if two programs can be made observationally equivalent for some lifting then they can be made equivalent for the lifting computed by $\rightharpoonup$. In other words, $\rightharpoonup$ is *complete.*

Theorem 4. *Let V_1', V_2', $\overline{V_1}$, and $\overline{V_2}$ be in $\boldsymbol{Com}_{\boldsymbol{\mathcal{V}}}$ and let $V_1, V_2 \in \boldsymbol{Com}$.*

1. *If $V_i \rightharpoonup \overline{V_i}$ can be derived then $\overline{V_i}$ is a lifting of V_i ($i = 1, 2$).*
2. *Suppose $\overline{V_1}$ ($\overline{V_2}$) shares no meta-variables with V_1', V_2', and $\overline{V_2}$ (V_1', V_2', and $\overline{V_1}$). If $V_1 \rightharpoonup \overline{V_1}$ and $V_2 \rightharpoonup \overline{V_2}$ can be derived and V_1' and V_2' are liftings*

of V_1, V_2, *respectively, then* $\mathcal{U}(\{V_1' \simeq^?_L V_2'\}) \neq \emptyset$ *implies* $\mathcal{U}(\{\overline{V_1} \simeq^?_L \overline{V_2}\}) \neq \emptyset$. *Furthermore,* $\mathcal{U}(\{V_1' \simeq^?_L V_1'\}) \neq \emptyset$ *implies* $\mathcal{U}(\{\overline{V_1} \simeq^?_L \overline{V_1}\}) \neq \emptyset$.

Integrating Standard Unification Algorithms. Standard algorithms for unification modulo an associative and commutative operator with neutral element and constants (see, e.g., [BS01] for background information on AC_1 unification) build on a characterization of equality that is equivalent to the one in Lemma 1. This correspondence allows one to employ existing algorithms for AC_1-unification problems with constants and free function symbols (like, e.g., the one in [HS87]) to the unification problems that arise when applying the rule for conditionals and then to filter the output such that only preserving substitutions remain.[7]

Automating Unification. In the following, we go beyond simply applying an existing unification algorithm by exploiting the specific shape of our unification problems and the limited range of substitutions in the computation of unifiers. Recall that we operate on programs in $Slice_{\mathcal{V}}$, i.e., on programs without assignments to high variables, without assignments of high expressions to low variables, and without loops or conditionals having high guards.

The operative intuition behind our problem-tailored unification algorithm is to scan two program terms from left to right and distinguish two cases: if both leftmost subcommands are free constructors, (low assignments, loops, conditionals and forks) they are compared and, if they agree, unification is recursively applied to pairs of corresponding subprograms and the residual programs. If one leftmost subcommand is skip, both programs are decomposed into their maximal initial subprograms in $Stut_{\mathcal{V}}$ and the remaining program. Unification is recursively applied to the corresponding subprograms. Formally, we define the language $NSeq_{\mathcal{V}}$ of commands in $Slice_{\mathcal{V}} \setminus \{\mathsf{skip}\}$ without sequential composition as a top-level operator, and the language $NStut_{\mathcal{V}}$ of commands in which the leftmost subcommand is not an element of $Stut_{\mathcal{V}}$. $NStut_{\mathcal{V}}$ is given by $C ::= C_1; C_2$, where $C_1 \in NSeq_{\mathcal{V}}$ and $C_2 \in Slice_{\mathcal{V}}$.

The unification algorithm in Figure 4 is given in form of a calculus for deriving judgments of the form $C_1 \simeq^?_L C_2 :: \eta$, meaning that η is a preserving unifier of the commands C_1 and C_2. Note that the unifiers obtained from recursive application of the algorithm to sub-programs are combined by set union. This is admissible if the meta-variables in all subprograms are disjoint, as the following lemma shows:

Lemma 2. *Let* $V_1, V_2 \in \boldsymbol{Slice_{\mathcal{V}}}$ *and let every variable occur at most once in* (V_1, V_2). *Then* $V_1 \simeq^?_L V_2 :: \eta$ *implies* $\eta \in \mathcal{U}(\{V_1 \simeq^?_L V_2\})$

Observe that the stand-alone unification algorithm is not complete, as it relies on the positions of meta-variables inserted by $\rightharpoonup$. However, we can prove a completeness result for the combination of both calculi.

[7] For the reader familiar with AC_1 unification: In the language $Stut_{\mathcal{V}}$ one views ϵ as the neutral element, skip as the constant, and ; as the operator. For $Slice_{\mathcal{V}}$, the remaining language constructs, i.e., assignments, conditionals, loops, forks, and ; (outside the language $Stut_{\mathcal{V}}$) must be treated as free constructors.

$$\frac{C_1 \simeq^?_L C_2 :: \eta \quad C_1, C_2 \in Stut_{\mathcal{V}}}{X; C_1 \simeq^?_L C_2 :: \eta[X \backslash \epsilon]} [Seq_1] \quad \frac{C_1 \simeq^?_L C_2 :: \eta \quad C_1, C_2 \in Stut_{\mathcal{V}}}{C_1 \simeq^?_L X; C_2 :: \eta[X \backslash \epsilon]} [Seq'_1]$$

$$\frac{C_1 \simeq^?_L C_2 :: \eta \quad C_1, C_2 \in Stut_{\mathcal{V}}}{\mathsf{skip}; C_1 \simeq^?_L \mathsf{skip}; C_2 :: \eta} [Seq_2] \quad \frac{C \in Stut_{\mathcal{V}} \cup \{\epsilon\}}{X \simeq^?_L C :: \{X \backslash C\}} [Var_1] \quad \frac{C \in Stut_{\mathcal{V}} \cup \{\epsilon\}}{C \simeq^?_L X :: \{X \backslash C\}} [Var_2]$$

$$\frac{C_1 \simeq^?_L C'_1 :: \eta_1 \quad C_2 \simeq^?_L C'_2 :: \eta_2 \quad C_1, C'_1 \in NSeq_{\mathcal{V}}}{C_1; C_2 \simeq^?_L C'_1; C'_2 :: \eta_1 \cup \eta_2} [Seq_3]$$

$$\frac{C_1 \simeq^?_L C'_1 :: \eta_1 \quad C_2 \simeq^?_L C'_2 :: \eta_2 \quad C_1, C'_1 \in Stut_{\mathcal{V}} \cup \{\epsilon\}, C_2, C'_2 \in NStut_{\mathcal{V}}}{C_1; C_2 \simeq^?_L C'_1; C'_2 :: \eta_1 \cup \eta_2} [Seq_4]$$

$$\frac{Id : low \quad Exp_1 \equiv Exp_2}{Id{:=}Exp_1 \simeq^?_L Id{:=}Exp_2 :: \emptyset} [Asg] \quad \frac{C \simeq^?_L C' :: \eta \quad V \simeq^?_L V' :: \eta_2}{\mathsf{fork}(CV) \simeq^?_L \mathsf{fork}(C'V') :: \eta_1 \cup \eta_2} [Frk]$$

$$\frac{C_1 \simeq^?_L C_2 :: \eta \quad B_1, B_2 : low \quad B_1 \equiv B_2}{\mathsf{while}\ B_1\ \mathsf{do}\ C_1 \simeq^?_L \mathsf{while}\ B_2\ \mathsf{do}\ C_2 :: \eta} [Whl] \quad \frac{C_1 \simeq^?_L C'_1 :: \eta_1, \ldots, C_n \simeq^?_L C'_n :: \eta_n}{\langle C_1, \ldots, C_n \rangle \simeq^?_L \langle C'_1, \ldots, C'_n \rangle :: \bigcup_{i=1}^n \eta_i} [Par]$$

$$\frac{C_1 \simeq^?_L C'_1 :: \eta_1 \quad C_2 \simeq^?_L C'_2 :: \eta_2 \quad B_1, B_2 : low \quad B_1 \equiv B_2}{\mathsf{if}\ B_1\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \simeq^?_L \mathsf{if}\ B_2\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2 :: \eta_1 \cup \eta_2} [Ite]$$

Fig. 4. Unification calculus

Completeness. If conditionals with high guards are nested then the process of transformational typing possibly involves repeated applications of substitutions to a given subprogram. Hence, care must be taken in choosing a substitution in each application of rule $[Cond_h]$ because, otherwise, unification problems in later applications of $[Cond_h]$ might become unsolvable.[8] Fortunately, the instantiation of our framework presented in this section does not suffer from such problems.

Theorem 5 (Completeness). *Let* $V \in \boldsymbol{Com}$, $\overline{V}, W \in \boldsymbol{Com}_{\mathcal{V}}$, *$W$ be a lifting of V, and* $V \rightharpoonup \overline{V}$.

1. *If there is a preserving substitution σ with $\sigma W \simeq_L \sigma W$, then $\overline{V} \hookrightarrow' V' : S$ for some $V', S \in \boldsymbol{Com}_{\mathcal{V}}$.*
2. *If $W \hookrightarrow W' : S$ for some $W', S \in \boldsymbol{Com}_{\mathcal{V}}$ then $\overline{V} \hookrightarrow' V' : S'$ for some $V', S' \in \boldsymbol{Com}_{\mathcal{V}}$.*

Here, the judgment $V \hookrightarrow' V' : S$ denotes a successful transformation of V to V' by the transformational type system, where the precondition $\sigma \in \mathcal{U}(\{S_1 \simeq^?_L S_2\})$ is replaced by $S_1 \simeq^?_L S_2 :: \sigma$ in rule $[Cond_h]$.

6 Related Work and Discussion

Type-based approaches to analyzing the security of the information flow in concrete programs have received much attention in recent years [SM03]. This

[8] A standard solution would be to apply *most general unifiers*. Unfortunately, they do not exist in our setting

resulted in security type systems for a broad range of languages (see, e.g., [VS97,SV98,HR98,Mye99,Sab01,SM02,BN02,HY02,BC02,ZM03,MS04]).

Regarding the analysis of conditionals with high guards, Volpano and Smith [VS98] proposed the atomic execution of entire conditionals for enforcing observational equivalence of alternative execution paths. This somewhat restrictive constraint is relaxed in the work of Agat [Aga00] and Sabelfeld and Sands [SS00] who achieve observational equivalence by cross-copying the slices of branches. The current article introduces unification modulo an equivalence relation as another alternative for making the branches of a conditional observationally equivalent to each other. Let us compare the latter two approaches more concretely for the relation $\simeq_L$ that we have introduced to instantiate our approach.

The type system introduced in Section 4 is capable of analyzing programs where assignments to low variables appear in the branches of conditionals with high guards, which is not possible with the type system in [SS00].

Example 1. If one lifts $C = \mathsf{if}\ h_1\ \mathsf{then}\ (h_2{:=}Exp_1; l{:=}Exp_2)\ \mathsf{else}\ (l{:=}Exp_2)$ where $Exp_2 : low$ using our lifting calculus, applies our transforming type system, and finally removes all remaining meta-variables by applying a projection then this results in $\mathsf{if}\ h_1\ \mathsf{then}\ (h_2{:=}Exp_1; l{:=}Exp_2)\ \mathsf{else}\ (\mathsf{skip}; l{:=}Exp_2)$, a program that is strongly secure and also weakly bisimilar to C. Note that the program C cannot be repaired by applying the type system from [SS00]. ♢

Another advantage of our unification-based approach over the cross-copying technique is that the resulting programs are faster and smaller in size.

Example 2. The program $\mathsf{if}\ h\ \mathsf{then}\ (h_1{:=}Exp_1)\ \mathsf{else}\ (h_2{:=}Exp_2)$ is returned unmodified by our type system, while the type system from [SS00] transforms it into the bigger program $\mathsf{if}\ h\ \mathsf{then}\ (h_1{:=}Exp_1; \mathsf{skip})\ \mathsf{else}\ (\mathsf{skip}; h_2{:=}Exp_2)$. If one applies this type system a second time, one obtains an even bigger program, namely $\mathsf{if}\ h\ \mathsf{then}\ (h_1{:=}Exp_1; \mathsf{skip}; \mathsf{skip}; \mathsf{skip})\ \mathsf{else}\ (\mathsf{skip}; \mathsf{skip}; \mathsf{skip}; h_2{:=}Exp_2)$. In contrast, our type system realizes a transformation that is idempotent, i.e. the program resulting from the transformation remains unmodified under a second application of the transformation. ♢

Non-transforming security type systems for the two-level security policy can be used to also analyze programs under a policy with more domains. To this end, one performs multiple type checks where each type check ensures that no illegitimate information flow can occur into a designated domain. For instance, consider a three-domain policy with domains $\mathcal{D} = \{top, left, right\}$ where information may only flow from *left* and from *right* to *top*. To analyze a program under this policy, one considers all variables with label *top* and *left* as if labeled *high* in a first type check (ensuring that there is no illegitimate information flow to *right*) and, in a second type check, considers all variables with label *top* and *right* as if labeled *high*. There is no need for a third type check as all information may flow to *top*. When adopting this approach for transforming type systems, one must take into account that the guarantees established by the type check for one domain might not be preserved under the modifications caused by the transformation for another domain. Therefore, one needs to iterate the process until a fixpoint is reached for all security domains.

Example 3. For the three-level policy from above, the program $C = \mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; r{:=}r'; l{:=}l')\ \mathsf{else}\ (r{:=}r'; l{:=}l')$ (assuming $t, t' : top$, $r, r' : right$ and $l, l' : left$) is lifted to $\overline{C} = \mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; r{:=}r'; \alpha_1; l{:=}l'; \alpha_2)\ \mathsf{else}\ (r{:=}r'; \alpha_3; l{:=}l'; \alpha_4)$ and transformed into $\mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; r{:=}r'; l{:=}l')\ \mathsf{else}\ (r{:=}r'; \mathsf{skip}; l{:=}l')$ when analyzing security w.r.t. an observer with domain *left*. Lifting for *right* then results in $\mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; \alpha_1; r{:=}r'; l{:=}l'; \alpha_2)\ \mathsf{else}\ (\alpha_3; r{:=}r'; \mathsf{skip}; l{:=}l'; \alpha_4)$. Unification and projection gives $\mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; r{:=}r'; l{:=}l'; \mathsf{skip})\ \mathsf{else}\ (\mathsf{skip}; r{:=}r'; \mathsf{skip}; l{:=}l')$. Observe that this program is not secure any more from the viewpoint of a *left*–observer. Applying the transformation again for domain *left* results in the secure program $\mathsf{if}\ t\ \mathsf{then}\ (t{:=}t'; r{:=}r'; \mathsf{skip}; l{:=}l'; \mathsf{skip})\ \mathsf{else}\ (\mathsf{skip}; r{:=}r'; \mathsf{skip}; l{:=}l'; \mathsf{skip})$, which is a fixpoint of both transformations. ♢

Note that the idempotence of the transformation is a crucial prerequisite (but not a sufficient one) for the existence of a fixpoint and, hence, for the termination of such an iterative approach. As is illustrated in Example 2, the transformation realized by our type system is idempotent, whereas the transformation from [SS00] is not.

Another possibility to tackle multi-level security policies in our setting is to unify the branches of a conditional with guard of security level D' under the theory $\bigcap_{D \not\geq D'} \simeq_D$. An investigation of this possibility remains to be done.

The chosen instantiation of our approach preserves the program behavior in the sense of a weak bisimulation. Naturally, one can correct more programs if one is willing to relax this relationship between input and output of the transformation. For this reason, there are also some programs that cannot be corrected with our type system although they can be corrected with the type system in [SS00] (which assumes a weaker relationship between input and output).

Example 4. $\mathsf{if}\ h\ \mathsf{then}\ (\mathsf{while}\ l\ \mathsf{do}\ (h_1{:=}Exp))\ \mathsf{else}\ (h_2{:=}1)$ is rejected by our type system. The type system in [SS00] transforms it into the strongly secure program $\mathsf{if}\ h\ \mathsf{then}\ (\mathsf{while}\ l\ \mathsf{do}\ (h_1{:=}Exp); \mathsf{skip})\ \mathsf{else}\ (\mathsf{while}\ l\ \mathsf{do}\ (\mathsf{skip}); h_2{:=}1)$. Note that this program is not weakly bisimilar to the original program as the cross-copying of the while loop introduces possible non-termination. ♢

If one wishes to permit such transformations, one could, for instance, choose a simulation instead of the weak bisimulation when instantiating our approach. This would result in an extended range of substitutions beyond $Stut_{\mathcal{V}}$. For instance, to correct the program in Example 4, one needs to instantiate a meta-variable with a while loop. We are confident that, in such a setting, using our approach would even further broaden the scope of corrections while retaining the advantage of transformed programs that are comparably small and fast.

7 Conclusions

We proposed a novel approach to analyzing the security of information flow in concrete programs with the help of transforming security type systems where the key idea has been to integrate unification with typing rules. This yielded a very natural perspective on the problem of eliminating implicit information flow.

We instantiated our approach by defining a program equivalence capturing the behavioral equivalence to be preserved during the transformation and an observational equivalence capturing the perspective of a low-level attacker. This led to a novel transforming security type system and calculi for automatically inserting meta-variables into programs and for computing substitutions. We proved that the resulting analysis technique is sound and also provided a relative completeness result. The main advantages of our approach include that the precision of type checking is improved, that additional insecure programs can be corrected, and that the resulting programs are faster and smaller in size.

It will be interesting to see how our approach performs for other choices of the parameters like, e.g., observational equivalences that admit intentional declassification [MS04]). Another interesting possibility is to perform the entire information flow analysis and program transformation using unification without any typing rules, which would mean to further explore the possibilities of the PER model. Finally, it would be desirable to integrate our fully automatic transformation into an interactive framework for supporting the programmer in correcting insecure programs.

References

[Aga00] J. Agat. Transforming out Timing Leaks. In *Proceedings of the 27th ACM Symposium on Principles of Programming Languages*, pages 40–53, 2000.

[BC02] G. Boudol and I. Castellani. Noninterference for Concurrent Programs and Thread Systems. *Theoretical Computer Science*, 281:109–130, 2002.

[BN02] A. Banerjee and D. A. Naumann. Secure Information Flow and Pointer Confinement in a Java-like Language. In *Proceedings of the 15th IEEE Computer Security Foundations Workshop*, pages 253–270, Cape Breton, Nova Scotia, Canada, 2002.

[BS01] F. Baader and W. Snyder. Unification theory. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 8, pages 445–532. Elsevier Science, 2001.

[HR98] N. Heintze and J. G. Riecke. The SLam Calculus: Programming with Secrecy and Integrity. In *Proceedings of the 25th ACM Symposium on Principles of Programming Languages*, pages 365–377, 1998.

[HS87] A. Herold and J. Siekmann. Unification in Abelian Semigroups. *Journal of Automated Reasoning*, 3:247–283, 1987.

[HY02] K. Honda and N. Yoshida. A uniform type structure for secure information flow. In *Proceedings of the 29th ACM Symposium on Principles of Programming Languages*, pages 81–92. ACM Press, 2002.

[MS04] Heiko Mantel and David Sands. Controlled Declassification based on Intransitive Noninterference. In *Proceedings of the 2nd ASIAN Symposium on Programming Languages and Systems, APLAS 2004*, LNCS 3303, pages 129–145, Taipei, Taiwan, November 4–6 2004. Springer-Verlag.

[Mye99] A. Myers. JFlow: Practical mostly-static information flow control. In *Symposium on Principles of Programming Languages*, pages 228–241, 1999.

[Sab01] A. Sabelfeld. The Impact of Synchronisation on Secure Information Flow in Concurrent Programs. In *Proceedings of Andrei Ershov 4th International Conference on Perspectives of System Informatics*, volume 2244 of *LNCS*, pages 225–239, 2001.

[SM02] A. Sabelfeld and H. Mantel. Static Confidentiality Enforcement for Distributed Programs. In *Proceedings of the 9th International Static Analysis Symposium, SAS'02*, volume 2477 of *LNCS*, pages 376–394, Madrid, Spain, 2002.

[SM03] A. Sabelfeld and A. C. Myers. Language-based Information-Flow Security. *IEEE Journal on Selected Areas in Communication*, 21(1):5–19, 2003.

[SS99] A. Sabelfeld and D. Sands. A Per Model of Secure Information Flow in Sequential Programs. In *Proceedings of the 8th European Symposium on Programming*, LNCS, pages 50–59, 1999.

[SS00] A. Sabelfeld and D. Sands. Probabilistic Noninterference for Multi-threaded Programs. In *Proceedings of the 13th IEEE Computer Security Foundations Workshop*, pages 200–215, Cambridge, UK, 2000.

[SV98] G. Smith and D. Volpano. Secure Information Flow in a Multi-Threaded Imperative Language. In *25th ACM Symposium on Principles of Programming Languages, San Diego, California*, pages 355–364, 1998.

[VS97] D. Volpano and G. Smith. A Type-Based Approach to Program Security. In *TAPSOFT 97*, volume 1214 of *LNCS*, pages 607–621, 1997.

[VS98] D. Volpano and G. Smith. Probabilistic Noninterference in a Concurrent Language. In *Proceedings of the 11th IEEE Computer Security Foundations Workshop*, pages 34–43, Rockport, Massachusetts, 1998.

[ZM03] S. Zdancewic and A. Myers. Observational determinism for concurrent program security. In *Proceedings of the 16th IEEE Computer Security Foundations Workshop, 2003*, pages 29–47. IEEE Computer Society, 2003.

A Semantics of MWL

The operational semantics for MWL are given in Figures 5 and 6.

B Proofs of the Technical Results

B.1 Proof of Theorem 1

Before proving Theorem 1, we introduce a lemma and prove it using the bisimulation-up-to technique.

Definition 7. *A binary relation R on commands is a* strong low bisimulation up to $\approxeq_L$ *if R is symmetric and*

$$\begin{array}{l}\forall C, C', C_1, \ldots, C_n \in Com\colon \forall s, s', t \in S\colon \\ \quad (C\ R\ C' \wedge s =_L s' \wedge \langle\!| C, s |\!\rangle \twoheadrightarrow \langle\!| C_1 \ldots C_n, t |\!\rangle) \\ \quad \Rightarrow \exists C'_1, \ldots, C'_n \in Com\colon \exists t' \in S\colon (\langle\!| C', s' |\!\rangle \twoheadrightarrow \langle\!| C'_1 \ldots C'_n, t' |\!\rangle \\ \qquad\qquad \wedge \forall i \in \{1, \ldots, n\} : C_i (R \cup \approxeq_L)^{+} C'_i \wedge t =_L t')\end{array}$$

$$\frac{\langle\!\langle C_i, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle W', t\rangle\!\rangle}{\langle\!\langle\langle C_0 \ldots C_{n-1}\rangle, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle C_0 \ldots C_{i-1}\rangle W'\langle C_{i+1} \ldots C_{n-1}\rangle, t\rangle\!\rangle}$$

Fig. 5. Small-step nondeterministic semantics

$$\langle\!\langle \mathsf{skip}, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle\rangle, s\rangle\!\rangle \qquad \frac{\langle\!\langle Exp, s\rangle\!\rangle \downarrow n}{\langle\!\langle Id{:=}Exp, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle\rangle, [Id = n]s\rangle\!\rangle}$$

$$\frac{\langle\!\langle C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle\rangle, t\rangle\!\rangle}{\langle\!\langle C_1; C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_2, t\rangle\!\rangle} \quad \frac{\langle\!\langle C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle C_1'\rangle V, t\rangle\!\rangle}{\langle\!\langle C_1; C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle C_1'; C_2\rangle V, t\rangle\!\rangle} \quad \langle\!\langle \mathsf{fork}(CV), s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle C\rangle V, s\rangle\!\rangle$$

$$\frac{\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{True}}{\langle\!\langle \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_1, s\rangle\!\rangle} \quad \frac{\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{False}}{\langle\!\langle \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_2, s\rangle\!\rangle}$$

$$\frac{\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{True}}{\langle\!\langle \mathsf{while}\ B\ \mathsf{do}\ C, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C; \mathsf{while}\ B\ \mathsf{do}\ C, s\rangle\!\rangle} \quad \frac{\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{False}}{\langle\!\langle \mathsf{while}\ B\ \mathsf{do}\ C, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle\langle\rangle, s\rangle\!\rangle}$$

Fig. 6. Small-step deterministic semantics

Theorem 6. *If R is a strong low bisimulation up to $\approxeq_L$ then $R \subseteq \approxeq_L$ holds.*

Proof (Sketch). Define $Q = ((R \cup \approxeq_L)^{\uparrow})^{+}$ where $^{+}$ returns the transitive closure of a relation and $^{\uparrow}$ returns the pointwise lifting of a relation on commands to command vectors (here, $R \cup \approxeq_L$ is viewed as a relation on commands). We obtain $Q \subseteq \approxeq_L$ from the fact that Q satisfies the implication in Definition 1 (replacing R in the definition by Q). The implication can be proved by a straightforward induction over the Q-distance between the related programs where the *Q-distance* between V and V' is the minimal length of a sequence $\langle V_1, \ldots, V_n\rangle$ with the properties $\forall i \in \{0, \ldots, n\} : V_i\ (R \cup \approxeq_L)^{\uparrow}\ V_{i+1}$, $V_0 = V$, and $V_{n+1} = V'$. Such a sequence exists because $V_0\ Q\ V_{n+1}$ holds. We obtain $R \subseteq \approxeq_L$ from $R \subseteq Q$ (by definition of Q) and the transitivity of $\subseteq$.

Lemma 3.

1. *If Id : high then $\mathsf{skip} \approxeq_L Id{:=}Exp$.*
2. *If Exp, Exp' : low and $Exp \equiv Exp'$ then $Id{:=}Exp \approxeq_L Id{:=}Exp'$.*
3. *If $C_1 \approxeq_L C_1'$ and $C_2 \approxeq_L C_2'$ then $C_1; C_2 \approxeq_L C_1'; C_2'$.*
4. *If $C_1 \approxeq_L C_1'$ and $V_2 \approxeq_L V_2'$ then $\mathsf{fork}(C_1 V_2) \approxeq_L \mathsf{fork}(C_1' V_2')$.*
5. *If B, B' : low, $B \equiv B'$, and $C_1 \approxeq_L C_1'$ then $\mathsf{while}\ B\ \mathsf{do}\ C_1 \approxeq_L \mathsf{while}\ B'\ \mathsf{do}\ C_1'$.*
6. *If B, B' : low, $B \equiv B'$, $C_1 \approxeq_L C_1'$, and $C_2 \approxeq_L C_2'$ then $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'$.*
7. *If $C_1 \approxeq_L C_1'$ and $C_1 \approxeq_L C_2'$ then $\mathsf{skip}; C_1 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'$.*

Proof. In each case, we prove the strong low bisimilarity of the two commands with the bisimulation up-to technique. That is, we define a binary relation R on commands that relates the two commands and prove that R is a strong low bisimulation up to $\approxeq_L$. From Theorem 6, we then obtain that the two given commands are strongly low bisimilar.

1. Define R as the symmetric closure of the relation $\{(\mathsf{skip}\ ,\ Id{:=}Exp) \mid Id : high\}$.
   Let $(\mathsf{skip}\ ,\ Id{:=}Exp) \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$. From the operational semantics, we obtain $\langle\!|\mathsf{skip}, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, s|\!\rangle$. Moreover, there is a $t' \in S$ such that $\langle\!|Id{:=}Exp, s'|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t'|\!\rangle$. From $s =_L s'$ and $Id : high$, we obtain $s =_L t'$.
   Let $(Id{:=}Exp\ ,\ \mathsf{skip}) \in R$ and $s, s', t \in S$ be arbitrary with $s =_L s'$ and $\langle\!|Id{:=}Exp, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t|\!\rangle$. We have $\langle\!|\mathsf{skip}, s'|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, s'|\!\rangle$. From $s =_L s'$ and $Id : high$, we obtain $t =_L s'$.
   Hence, R is a strong low bisimulation up to $\approxeq_L$.
2. Define R as the symmetric relation $\{(Id{:=}Exp\ ,\ Id{:=}Exp') \mid Exp, Exp' : low, Exp \equiv Exp'\}$.
   Let $(Id{:=}Exp\ ,\ Id{:=}Exp') \in R$ and $s, s', t \in S$ be arbitrary with $s =_L s'$ and $\langle\!|Id{:=}Exp, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t|\!\rangle$. Choose $t' \in S$ with $\langle\!|Id{:=}Exp', s'|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t'|\!\rangle$. From $s =_L s'$, $Exp, Exp' : low$, and $Exp \equiv Exp'$, we obtain $t =_L t'$.
   Hence, R is a strong low bisimulation up to $\approxeq_L$.
3. Define R as the symmetric relation $\{(C_1; C_2\ ,\ C_1'; C_2') \mid C_1 \approxeq_L C_1', C_2 \approxeq_L C_2'\}$.
   Let $(C_1; C_2\ ,\ C_1'; C_2') \in R$ and $s, s', t \in S$ be arbitrary with $s =_L s'$ and $\langle\!|C_1; C_2, s|\!\rangle \twoheadrightarrow \langle\!|C^*, t|\!\rangle$ for some $C^* \in \boldsymbol{Com}$. We make a case distinction on C^* according to the operational semantics:
   (a) $C^* = C_2$: From the operational semantics, we obtain $\langle\!|C_1, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t|\!\rangle$. Since $C_1 \approxeq_L C_1'$ and $s =_L s'$, there is a $t' \in S$ with $\langle\!|C_1', s'|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t'|\!\rangle$ and $t =_L t'$ according to Definition 1. From the operational semantics, we obtain $\langle\!|C_1'; C_2', s'|\!\rangle \twoheadrightarrow \langle\!|C_2', t'|\!\rangle$ with $C_2 \approxeq_L C_2'$ (by definition of R) and $t =_L t'$.
   (b) $C^* = \langle C; C_2\rangle V$ for some $C \in Com$ and $V \in \boldsymbol{Com}$ (possibly $V = \langle\rangle$). From the operational semantics, we obtain $\langle\!|C_1, s|\!\rangle \twoheadrightarrow \langle\!|\langle C\rangle V, t|\!\rangle$. Since, $C_1 \approxeq_L C_1'$ and $s =_L s'$, there are $C' \in Com$, $V' \in \boldsymbol{Com}$, and $t' \in S$ with $\langle\!|C_1', s'|\!\rangle \twoheadrightarrow \langle\!|\langle C'\rangle V', t'|\!\rangle$, $C \approxeq_L C'$, $V \approxeq_L V'$, and $t =_L t'$. From the operational semantics, we obtain $\langle\!|C_1'; C_2', s'|\!\rangle \twoheadrightarrow \langle\!|\langle C'; C_2'\rangle V', t'|\!\rangle$ with $(C; C_2\ ,\ C'; C_2') \in R$ (follows from $C \approxeq_L C'$, $C_2 \approxeq_L C_2'$, and the definition of R), $V \approxeq_L V'$, and $t =_L t'$.
   Hence, R is a strong low bisimulation up to $\approxeq_L$.
4. Define R as the symmetric relation $\{(\mathsf{fork}(CV)\ ,\ \mathsf{fork}(C'V')) \mid C \approxeq_L C', V \approxeq_L V'\}$.
   Let $(\mathsf{fork}(CV)\ ,\ \mathsf{fork}(C'V')) \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$. From the operational semantics, we obtain $\langle\!|\mathsf{fork}(CV), s|\!\rangle \twoheadrightarrow \langle\!|\langle C\rangle V, s|\!\rangle$ and $\langle\!|\mathsf{fork}(C'V'), s'|\!\rangle \twoheadrightarrow \langle\!|\langle C'\rangle V', s'|\!\rangle$. From the definition of R and Definition 1, we obtain $\langle C\rangle V \approxeq_L \langle C'\rangle V'$.
   Hence, R is a strong low bisimulation up to $\approxeq_L$.
5. Define R as the symmetric relation

$$\begin{array}{l}\{\ (\mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ \mathsf{while}\ B'\ \mathsf{do}\ C_1'), (C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_2') \\ \ \mid\ B, B' : low, B \equiv B', C_1 \approxeq_L C_1', C_2 \approxeq_L C_2'\}\end{array}$$

   Let $(\mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ \mathsf{while}\ B'\ \mathsf{do}\ C_1') \in R$ and $s, s', t \in S$ be arbitrary with $s =_L s'$. We make a case distinction on the value of B in s:

(a) $\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{False}$: From $s =_L s'$, $B, B' : low$, and $B \equiv B'$, we obtain $\langle\!\langle B', s'\rangle\!\rangle \downarrow \mathsf{False}$. From the operational semantics, we obtain $\langle\!\langle \mathsf{while}\ B\ \mathsf{do}\ C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle\rangle, s\rangle\!\rangle$ and $\langle\!\langle \mathsf{while}\ B'\ \mathsf{do}\ C_1', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle\rangle, s'\rangle\!\rangle$ with $s =_L s'$.

(b) $\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{True}$: From $s =_L s'$, $B, B' : low$, and $B \equiv B'$, we obtain $\langle\!\langle B', s'\rangle\!\rangle \downarrow \mathsf{True}$. From the operational semantics, we obtain $\langle\!\langle \mathsf{while}\ B\ \mathsf{do}\ C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_1; \mathsf{while}\ B\ \mathsf{do}\ C_1, s\rangle\!\rangle$ and $\langle\!\langle \mathsf{while}\ B'\ \mathsf{do}\ C_1', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_1', s'\rangle\!\rangle$ with $(C_1; \mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_1') \in R$ (follows from $B, B' : low$, $B \equiv B'$, $C_1 \approxeq_L C_1'$, and the definition of R) and $s =_L s'$.

Let $(C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_2') \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$ and $\langle\!\langle C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C^*, t\rangle\!\rangle$ for some $C^* \in \boldsymbol{Com}$. We make a case distinction on C^* according to the operational semantics:

(a) $C^* = \mathsf{while}\ B\ \mathsf{do}\ C_2$: From the operational semantics, we obtain $\langle\!\langle C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle\rangle, t\rangle\!\rangle$. Since $C_1 \approxeq_L C_1'$ and $s =_L s'$, there is a $t' \in S$ with $\langle\!\langle C_1', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle\rangle, t'\rangle\!\rangle$ and $t =_L t'$ according to Definition 1. From the operational semantics, we obtain $\langle\!\langle C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_2', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle \mathsf{while}\ B'\ \mathsf{do}\ C_2', t'\rangle\!\rangle$ with $(\mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ \mathsf{while}\ B'\ \mathsf{do}\ C_2') \in R$ (follows from $B, B' : low$, $B \equiv B'$, $C_2 \approxeq_L C_2'$, and the definition of R) and $t =_L t'$.

(b) $C^* = \langle C; \mathsf{while}\ B\ \mathsf{do}\ C_2\rangle V$ for some $C \in Com$ and $V \in \boldsymbol{Com}$ (possibly $V = \langle\rangle$). From the operational semantics, we obtain $\langle\!\langle C_1, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle C\rangle V, t\rangle\!\rangle$. Since, $C_1 \approxeq_L C_1'$ and $s =_L s'$, there are $C' \in Com$, $V' \in \boldsymbol{Com}$, and $t' \in S$ with $\langle\!\langle C_1', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle C'\rangle V', t'\rangle\!\rangle$, $C \approxeq_L C'$, $V \approxeq_L V'$, and $t =_L t'$. From the operational semantics, we obtain $\langle\!\langle C_1'; \mathsf{while}\ B'\ \mathsf{do}\ C_2', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle \langle C'; \mathsf{while}\ B'\ \mathsf{do}\ C_2'\rangle V', t'\rangle\!\rangle$ with $(C; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C'; \mathsf{while}\ B'\ \mathsf{do}\ C_2') \in R$ (follows from $C \approxeq_L C'$, $\mathsf{while}\ B\ \mathsf{do}\ C_2 \approxeq_L \mathsf{while}\ B'\ \mathsf{do}\ C_2'$, and the definition of R), $V \approxeq_L V'$, and $t =_L t'$.

Hence, R is a strong low bisimulation up to $\approxeq_L$.

6. Define R as the symmetric relation $\{(\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2\ ,\ \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2') \mid B, B' : low, B \equiv B', C_1 \approxeq_L C_1', C_2 \approxeq_L C_2'\}$.

   Let $(\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2\ ,\ \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2') \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$. We make a case distinction on the value of B in s:

   (a) $\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{False}$: From $s =_L s'$, $B, B' : low$, and $B \equiv B'$, we obtain $\langle\!\langle B', s'\rangle\!\rangle \downarrow \mathsf{False}$. From the operational semantics, we obtain $\langle\!\langle \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_2, s\rangle\!\rangle$ and $\langle\!\langle \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_2', s'\rangle\!\rangle$ with $s =_L s'$. By definition of R, we have $C_2 \approxeq_L C_2'$.

   (b) $\langle\!\langle B, s\rangle\!\rangle \downarrow \mathsf{True}$: From $s =_L s'$, $B, B' : low$, and $B \equiv B'$, we obtain $\langle\!\langle B', s'\rangle\!\rangle \downarrow \mathsf{True}$. From the operational semantics, we obtain $\langle\!\langle \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_1, s\rangle\!\rangle$ and $\langle\!\langle \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s'\rangle\!\rangle \twoheadrightarrow \langle\!\langle C_1', s'\rangle\!\rangle$ with $s =_L s'$. By definition of R, we have $C_1 \approxeq_L C_1'$.

   Hence, R is a strong low bisimulation up to $\approxeq_L$.

7. Define R as the symmetric closure of the relation

$$\{(\mathsf{skip}; C_1\ ,\ \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2') \mid C_1 \approxeq_L C_1', C_1 \approxeq_L C_2'\}.$$

   Let $(\mathsf{skip}; C_1\ ,\ \mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2') \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$. We make a case distinction on the value of B' in s':

(a) $\langle\!|B', s'|\!\rangle \downarrow \mathsf{False}$: From the operational semantics, we obtain $\langle\!|\mathsf{skip}; C_1, s|\!\rangle \twoheadrightarrow \langle\!|C_1, s|\!\rangle$ and $\langle\!|\mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s'|\!\rangle \twoheadrightarrow \langle\!|C_2', s'|\!\rangle$ with $s =_L s'$. By definition of R, we have $C_1 \approxeq_L C_2'$.

(b) $\langle\!|B', s'|\!\rangle \downarrow \mathsf{True}$: From the operational semantics, we obtain $\langle\!|\mathsf{skip}; C_1, s|\!\rangle \twoheadrightarrow \langle\!|C_1, s|\!\rangle$ and $\langle\!|\mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s'|\!\rangle \twoheadrightarrow \langle\!|C_1', s'|\!\rangle$ with $s =_L s'$. By definition of R, we have $C_1 \approxeq_L C_1'$.

Let $(\mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'\ ,\ \mathsf{skip}; C_1) \in R$ and $s, s' \in S$ be arbitrary with $s =_L s'$. We make a case distinction on the value of B' in s:

(a) $\langle\!|B', s|\!\rangle \downarrow \mathsf{False}$: Then we obtain $\langle\!|\mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s|\!\rangle \twoheadrightarrow \langle\!|C_2', s|\!\rangle$ and $\langle\!|\mathsf{skip}; C_1, s'|\!\rangle \twoheadrightarrow \langle\!|C_1, s'|\!\rangle$ from the operational semantics with $s =_L s'$. By definition of R, we have $C_1 \approxeq_L C_2'$.

(b) $\langle\!|B', s|\!\rangle \downarrow \mathsf{True}$: Then we obtain $\langle\!|\mathsf{if}\ B'\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2', s|\!\rangle \twoheadrightarrow \langle\!|C_1', s|\!\rangle$ and $\langle\!|\mathsf{skip}; C_1, s'|\!\rangle \twoheadrightarrow \langle\!|C_1, s'|\!\rangle$ from the operational semantics with $s =_L s'$. By definition of R, we have $C_1 \approxeq_L C_1'$.

Hence, R is a strong low bisimulation up to $\approxeq_L$.

We are now ready to prove Theorem 1.

Proof (of Theorem 1). The proof proceeds by induction on the number of rule applications in the derivation $\mathcal{D}$ of $V \simeq_L V'$.

Base case: $\mathcal{D}$ consists of only a single rule application. We make a case distinction on this rule.

[*Skip*] The judgment derived is $\mathsf{skip} \simeq_L \mathsf{skip}$. Lemma 3(1) implies $\mathsf{skip} \approxeq_L h{:=}h$ where h is an arbitrary variable with $h : \mathit{high}$. From symmetry and transitivity of $\approxeq_L$, we obtain $\mathsf{skip} \approxeq_L \mathsf{skip}$.

[SHA_1] The judgment derived is $\mathsf{skip} \simeq_L \mathit{Id}{:=}\mathit{Exp}$ with $\mathit{Id} : \mathit{high}$. From Lemma 3(1) follows $\mathsf{skip} \approxeq_L \mathit{Id}{:=}\mathit{Exp}$.

[SHA_2] The judgment derived is $\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathsf{skip}$ with $\mathit{Id} : \mathit{high}$. From Lemma 3(1) and the symmetry of $\approxeq_L$, we obtain $\mathit{Id}{:=}\mathit{Exp} \approxeq_L \mathsf{skip}$.

[*HA*] The judgment derived is $\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathit{Id}'{:=}\mathit{Exp}'$ with $\mathit{Id}, \mathit{Id}' : \mathit{high}$. From Lemma 3(1), we obtain $\mathsf{skip} \approxeq_L \mathit{Id}{:=}\mathit{Exp}$ and $\mathsf{skip} \approxeq_L \mathit{Id}'{:=}\mathit{Exp}'$. Symmetry and transitivity of $\approxeq_L$ imply $\mathit{Id}{:=}\mathit{Exp} \approxeq_L \mathit{Id}'{:=}\mathit{Exp}'$.

[*LA*] The judgment derived is $\mathit{Id}{:=}\mathit{Exp} \simeq_L \mathit{Id}{:=}\mathit{Exp}'$ with $\mathit{Id} : \mathit{low}$, $\mathit{Exp}, \mathit{Exp}' : \mathit{low}$, and $\mathit{Exp} \equiv \mathit{Exp}'$. Lemma 3(2) implies $\mathit{Id}{:=}\mathit{Exp} \approxeq_L \mathit{Id}{:=}\mathit{Exp}'$.

Induction assumption: If $\mathcal{D}'$ is a derivation of $W \simeq_L W'$ with fewer rule applications than in $\mathcal{D}$ then $W \approxeq_L W'$ holds.

Step case: We make a case distinction depending on the rule applied at the root of $\mathcal{D}$.

[*SComp*] The judgment derived is $C_1; C_2 \simeq_L C_1'; C_2'$ and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \simeq_L C_1'$ and $C_2 \simeq_L C_2'$, respectively. From the induction assumption, we obtain $C_1 \approxeq_L C_1'$ and $C_2 \approxeq_L C_2'$. Lemma 3(3) implies $C_1; C_2 \approxeq_L C_1'; C_2'$.

[*PComp*] The judgment derived is $\langle C_1, \ldots, C_n \rangle \simeq_L \langle C_1', \ldots, C_n' \rangle$ and there are derivations $\mathcal{D}_i$ of $C_i \simeq_L C_i'$ for $i = 1, \ldots, n$. From the induction assumption, we obtain $C_i \approxeq_L C_i'$ for $i = 1, \ldots, n$. From Definition 1, we obtain $\langle C_1, \ldots, C_n \rangle \approxeq_L \langle C_1', \ldots, C_n' \rangle$.

[*Fork*] The judgment derived is $\mathsf{fork}(C_1 V_1) \simeq_L \mathsf{fork}(C_1' V_1')$ and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \simeq_L C_1'$ and $V_1 \simeq_L V_1'$, respectively. From the induction assumption, we obtain $C_1 \approxeq_L C'_1$ and $V_1 \approxeq_L V'_1$. Lemma 3(4) implies $\mathsf{fork}(C_1 V_1) \approxeq_L \mathsf{fork}(C'_1 V'_1)$.

[*WL*] The judgment derived is $\mathsf{while}\ B\ \mathsf{do}\ C_1 \doteq_L \mathsf{while}\ B'\ \mathsf{do}\ C'_1$ with $B, B' : low$, $B \equiv B'$, and there is a derivation $\mathcal{D}_1$ of $C_1 \doteq_L C'_1$. Lemma 3(5) implies $\mathsf{while}\ B\ \mathsf{do}\ C_1 \approxeq_L \mathsf{while}\ B'\ \mathsf{do}\ C'_1$.

[*LIte*] The judgment derived is $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \doteq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$ with $B, B' : low$, $B \equiv B'$, and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \doteq_L C'_1$ and $C_2 \doteq_L C'_2$, respectively. Lemma 3(6) implies $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$.

[*SHIte$_1$*] The judgment derived is $\mathsf{skip}; C_1 \doteq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$ with $B' : high$ and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \doteq_L C'_1$ and $C_1 \doteq_L C'_2$, respectively. From the induction assumption, we obtain $C_1 \approxeq_L C'_1$ and $C_1 \approxeq_L C'_2$. Lemma 3(7) implies $\mathsf{skip}; C_1 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$.

[*SHIte$_2$*] The judgment derived is $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \doteq_L \mathsf{skip}; C'_1$ with $B : high$ and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \doteq_L C'_1$ and $C_2 \doteq_L C'_1$, respectively. From the induction assumption, we obtain $C_1 \approxeq_L C'_1$ and $C_2 \approxeq_L C'_1$. Symmetry of $\approxeq_L$ and Lemma 3(7) imply $\mathsf{skip}; C'_1 \approxeq_L \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2$. From the symmetry of $\approxeq_L$, we obtain $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L \mathsf{skip}; C'_1$

[*HAHIte$_1$*] The judgment derived is $Id{:=}Exp; C_1 \doteq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$ with $Id : high$, $B' : high$, and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \doteq_L C'_1$ and $C_1 \doteq_L C'_2$, respectively. From $\mathsf{skip}; C_1 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$ (see Case [*SHIte$_1$*]), $Id{:=}Exp; C_1 \approxeq_L \mathsf{skip}; C_1$, and transitivity of $\approxeq_L$, we obtain $Id{:=}Exp; C_1 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$.

[*HAHIte$_2$*] The judgment derived is $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \doteq_L Id'{:=}Exp'; C'_1$ with $Id' : high$, $B : high$, and there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $C_1 \doteq_L C'_1$ and $C_2 \doteq_L C'_1$, respectively. From $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L \mathsf{skip}; C'_1$ (see Case [*SHIte$_2$*]), $\mathsf{skip}; C'_1 \approxeq_L Id'{:=}Exp'; C'_1$, and transitivity of $\approxeq_L$, we obtain $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L Id'{:=}Exp'; C'_1$.

[*HIte*] The judgment derived is $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \doteq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$ with $B, B' : high$ and there are derivations $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ of $C_1 \doteq_L C'_1$, $C_1 \doteq_L C'_2$, and $C_1 \doteq_L C_2$, respectively. From the induction assumption, we obtain $C_1 \approxeq_L C'_1$, $C_1 \approxeq_L C'_2$, and $C_1 \approxeq_L C_2$. From symmetry and transitivity of $\approxeq_L$ follows $C_1 \approxeq_L C_1$. Lemma 3(7) implies $\mathsf{skip}; C_1 \approxeq_L \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2$ and $\mathsf{skip}; C_1 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$. From the symmetry and transitivity of $\approxeq_L$, we obtain $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \approxeq_L \mathsf{if}\ B'\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$.

B.2 Proof of Theorem 2

For the proof of Theorem 2 we first strengthen our notion of bisimulation. Then we prove a lemma that shows that this relation is a congruence by using the up-to technique.

Definition 8. *The* pointwise weak bisimulation $\dot{\simeq}$ *is the union of all symmetric relations R on command vectors $V, V' \in$ **Com** of equal size, i.e. $V = \langle C_1, \ldots, C_n \rangle$ and $V' = \langle C'_1, \ldots, C'_n \rangle$, such that whenever $V R V'$ then for all* *states s, t and all $i \in \{1 \ldots n\}$ and all thread pools W there is a thread pool W' with*

$$\langle\!|C_i, s|\!\rangle \twoheadrightarrow \langle\!|W, t|\!\rangle) \Rightarrow (\langle\!|C'_i, s|\!\rangle \twoheadrightarrow^* \langle\!|W', t|\!\rangle \wedge W\, R\, W')$$
$$\text{and } V = \langle\rangle \implies \langle\!|V', s|\!\rangle \rightarrow^* \langle\!|\langle\rangle, s|\!\rangle.$$

Lemma 4. $V \mathrel{\dot{\simeq}} V' \Rightarrow V \simeq V'$

Proof. Follows directly from Definitions 2 and 8 and the operational semantics for thread pools.

Definition 9. *A binary relation R on commands is a* pointwise weak bisimulation up to $\dot{\simeq}$ *if R is symmetric and*

$$\begin{array}{l}\forall C, C', C_1, \ldots, C_n \in \mathit{Com}\colon \forall s, t \in S\colon \\ \quad (C\ R\ C' \wedge \langle\!|C, s|\!\rangle \twoheadrightarrow \langle\!|C_1 \ldots C_n, t|\!\rangle) \\ \quad \Rightarrow \exists C'_1, \ldots, C'_n \in \mathit{Com}\colon (\langle\!|C', s|\!\rangle \twoheadrightarrow^* \langle\!|C'_1 \ldots C'_n, t|\!\rangle \\ \qquad \wedge \forall i \in \{1, \ldots, n\} : C_i (R \cup \dot{\simeq}) C'_i).\end{array}$$

Theorem 7. *If R is a pointwise weak bisimulation up to $\dot{\simeq}$, then we have $R \subseteq \dot{\simeq}$.*

Proof. Let $Q = (R \cup \dot{\simeq})^{\uparrow}$, where $^{\uparrow}$ returns the pointwise lifting of a relation on commands to command vectors (here, $R \cup \dot{\simeq}$ is viewed as a relation on commands). It is sufficient to show $Q \subseteq \dot{\simeq}$. To this means, we show that Q satisfies the condition in Definition 8, and is therefore contained in $\dot{\simeq}$, the union of all such relations. Let $V = \langle C_1, \ldots, C_n\rangle$ and $V' = \langle C'_1, \ldots, C'_n\rangle$ and $(V, V') \in Q$. If $n = 0$, then the second condition of Definition 8 is fulfilled. Suppose $n > 0$ and $\langle\!|V, s|\!\rangle \twoheadrightarrow \langle\!|W, t|\!\rangle$. By definition of the operational semantics, we know that there is $i \in \{1, \ldots, n\}$ with $\langle\!|C_i, s|\!\rangle \twoheadrightarrow \langle\!|C_{i,1}, \ldots, C_{i,m}, t|\!\rangle$ and $W = \langle C_1, \ldots, C_{i-1}, C_{i,1}, \ldots, C_{i,m}, C_{i+1}, \ldots, C_n\rangle$.

1. *Case 1:* $(C_i, C'_i) \in \dot{\simeq}$. By Definition 8, there are $C'_{i,1}, \ldots, C'_{i,m}$ with $\langle\!|C'_i, s|\!\rangle \twoheadrightarrow^* \langle\!|C'_{i,1}, \ldots, C'_{i,m}, t|\!\rangle$ and $\langle C_{i,1}, \ldots, C_{i,m}\rangle \mathrel{\dot{\simeq}} \langle C'_{i,1}, \ldots, C'_{i,m}\rangle$. By the nature of Definition 8 it follows that $C_{i,j} \mathrel{\dot{\simeq}} C_{i,j}$ holds for all $j \in \{1, \ldots, m\}$.
2. *Case 2:* $(C_i, C'_i) \in R$. By Definition 9, there are $C'_{i,1}, \ldots, C'_{i,m}$ with $\langle\!|C'_i, s|\!\rangle \twoheadrightarrow^* \langle\!|C'_{i,1}, \ldots, C'_{i,m}, t|\!\rangle$ and $C_{i,j}(R \cup \dot{\simeq})C'_{i,j}$ for all $j \in \{1, \ldots, m\}$.

With $W' = \langle C'_1, \ldots, C'_{i-1}, C'_{i,1}, \ldots, C'_{i,m}, C'_{i+1}, \ldots, C'_n\rangle$ we have $\langle\!|V', s|\!\rangle \twoheadrightarrow^* \langle\!|W', t|\!\rangle$ and $(W, W') \in Q$. As $\dot{\simeq}$ is defined to be the union of all symmetric relations with the property of Definition 8, we see $Q \subseteq \dot{\simeq}$.

Lemma 5.

1. *If $C_1 \mathrel{\dot{\simeq}} C'_1$ and $C_2 \mathrel{\dot{\simeq}} C'_2$ then $C_1; C_2 \mathrel{\dot{\simeq}} C'_1; C'_2$.*

2. *If* $C_1 \mathrel{\dot{\simeq}} C_1'$ *and* $V_2 \mathrel{\dot{\simeq}} V_2'$ *then* $\mathsf{fork}(C_1V_2) \mathrel{\dot{\simeq}} \mathsf{fork}(C_1'V_2')$.
3. *If* $C_1 \mathrel{\dot{\simeq}} C_1'$ *then* $\mathsf{while}\ B\ \mathsf{do}\ C_1 \mathrel{\dot{\simeq}} \mathsf{while}\ B\ \mathsf{do}\ C_1'$.
4. *If* $C_1 \mathrel{\dot{\simeq}} C_1'$, *and* $C_2 \mathrel{\dot{\simeq}} C_2'$ *then* $\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2 \mathrel{\dot{\simeq}} \mathsf{if}\ B\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'$.
5. *If* $C_1 \mathrel{\dot{\simeq}} C_1', \ldots, C_n \mathrel{\dot{\simeq}} C_n'$, *then* $\langle C_1, \ldots, C_n\rangle \mathrel{\dot{\simeq}} \langle C_1', \ldots, C_n'\rangle$.

Proof. We proceed as in the proof of Lemma 3, only by using pointwise weak bisimulations up to $\dot{\simeq}$ instead of strong low-bisimulations.

1. Define R as the symmetric relation $\{(C_1; C_2\ ,\ C_1'; C_2') \mid C_1 \mathrel{\dot{\simeq}} C_1', C_2 \mathrel{\dot{\simeq}} C_2'\}$. Let $(C_1; C_2\ ,\ C_1'; C_2') \in R$ and $s, t \in S$ be arbitrary with $\langle\!|C_1; C_2, s|\!\rangle \twoheadrightarrow \langle\!|C^*, t|\!\rangle$ for some $C^* \in \boldsymbol{Com}$. We make a case distinction on C^* according to the operational semantics:
   (a) $C^* = C_2$: From the operational semantics, we see $\langle\!|C_1, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, t|\!\rangle$. Since $C_1 \mathrel{\dot{\simeq}} C_1'$, we have $\langle\!|C_1', s|\!\rangle \twoheadrightarrow^* \langle\!|C_1'', t|\!\rangle$ with $\langle\rangle \mathrel{\dot{\simeq}} \langle\!|C_1'', t|\!\rangle$. By the second part of Definition 2, we obtain $\langle\!|C_1'', t|\!\rangle \twoheadrightarrow^* \langle\!|\langle\rangle, t|\!\rangle$ and so we see $\langle\!|C_1', s|\!\rangle \twoheadrightarrow^* \langle\!|\langle\rangle, t|\!\rangle$. From the operational semantics, we thus see $\langle\!|C_1'; C_2', s|\!\rangle \twoheadrightarrow^* \langle\!|C_2', t|\!\rangle$ with $C_2 \mathrel{\dot{\simeq}} C_2'$ by definition of R.
   (b) $C^* = \langle C; C_2\rangle V$ for some $C \in Com$ and possibly empty $V \in \boldsymbol{Com}$. From the operational semantics, we obtain $\langle\!|C_1, s|\!\rangle \twoheadrightarrow \langle\!|\langle C\rangle V, t|\!\rangle$. Since $C_1 \mathrel{\dot{\simeq}} C_1'$, there are $C' \in Com$ and $V' \in \boldsymbol{Com}$ with $\langle\!|C_1', s|\!\rangle \twoheadrightarrow^* \langle\!|\langle C'\rangle V', t|\!\rangle$, $C \mathrel{\dot{\simeq}} C'$ and $V \mathrel{\dot{\simeq}} V'$. From the operational semantics we obtain $\langle\!|C_1'; C_2', s|\!\rangle \twoheadrightarrow^* \langle\!|\langle C'; C_2'\rangle V', t|\!\rangle$ with $(C; C_2\ ,\ C'; C_2') \in R$ (follows from $C \mathrel{\dot{\simeq}} C'$, $C_2 \mathrel{\dot{\simeq}} C_2'$, and the definition of R) and $V \mathrel{\dot{\simeq}} V'$.

   Hence, R is a pointwise weak bisimulation up to $\dot{\simeq}$.
2. Define R as the symmetric relation $\{(\mathsf{fork}(CV)\ ,\ \mathsf{fork}(C'V')) \mid C \mathrel{\dot{\simeq}} C', V \mathrel{\dot{\simeq}} V'\}$. Let $(\mathsf{fork}(CV)\ ,\ \mathsf{fork}(C'V')) \in R$ and $s \in S$ be arbitrary. From the operational semantics, we obtain $\langle\!|\mathsf{fork}(CV), s|\!\rangle \twoheadrightarrow \langle\!|\langle C\rangle V, s|\!\rangle$ and $\langle\!|\mathsf{fork}(C'V'), s|\!\rangle \twoheadrightarrow \langle\!|\langle C'\rangle V', s|\!\rangle$. From the definition of R and Definition 8, we obtain $\langle C\rangle V \mathrel{\dot{\simeq}} \langle C'\rangle V'$.

   Hence, R is a pointwise weak bisimulation up to $\dot{\simeq}$.
3. Define R as the symmetric relation

   $$\{\ (\mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ \mathsf{while}\ B\ \mathsf{do}\ C_1'), (C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C_1'; \mathsf{while}\ B\ \mathsf{do}\ C_2') \\ \mid\ C_1 \mathrel{\dot{\simeq}} C_1', C_2 \mathrel{\dot{\simeq}} C_2'\}$$

   Let $(\mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ \mathsf{while}\ B\ \mathsf{do}\ C_1') \in R$ and $s, t \in S$ be arbitrary. We make a case distinction on the value of B in s:
   (a) $\langle\!|B, s|\!\rangle \downarrow \mathsf{False}$: From the operational semantics, we get $\langle\!|\mathsf{while}\ B\ \mathsf{do}\ C_1, s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, s|\!\rangle$ and $\langle\!|\mathsf{while}\ B\ \mathsf{do}\ C_1', s|\!\rangle \twoheadrightarrow \langle\!|\langle\rangle, s|\!\rangle$.
   (b) $\langle\!|B, s|\!\rangle \downarrow \mathsf{True}$: From the operational semantics, we get $\langle\!|\mathsf{while}\ B\ \mathsf{do}\ C_1, s|\!\rangle \twoheadrightarrow \langle\!|C_1; \mathsf{while}\ B\ \mathsf{do}\ C_1, s|\!\rangle$ and $\langle\!|\mathsf{while}\ B\ \mathsf{do}\ C_1', s|\!\rangle \twoheadrightarrow \langle\!|C_1'; \mathsf{while}\ B\ \mathsf{do}\ C_1', s|\!\rangle$ with $(C_1; \mathsf{while}\ B\ \mathsf{do}\ C_1\ ,\ C_1'; \mathsf{while}\ B\ \mathsf{do}\ C_1') \in R$.

Let $(C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C'_1; \mathsf{while}\ B\ \mathsf{do}\ C'_2) \in R$ and $s \in S$ be arbitrary with $\langle\!| C_1; \mathsf{while}\ B\ \mathsf{do}\ C_2, s |\!\rangle \twoheadrightarrow \langle\!| C^*, t |\!\rangle$ for some $C^* \in \boldsymbol{Com}$. We make a case distinction on C^* according to the operational semantics:

(a) $C^* = \mathsf{while}\ B\ \mathsf{do}\ C_2$: From the operational semantics, we obtain $\langle\!| C_1, s |\!\rangle \twoheadrightarrow \langle\!| \langle\rangle, t |\!\rangle$. Since $C_1 \doteq C'_1$ we obtain $\langle\!| C'_1, s |\!\rangle \twoheadrightarrow^* \langle\!| \langle\rangle, t |\!\rangle$ similar to the sequential composition case. From the operational semantics, we obtain $\langle\!| C'_1; \mathsf{while}\ B\ \mathsf{do}\ C'_2, s |\!\rangle \twoheadrightarrow^* \langle\!| \mathsf{while}\ B\ \mathsf{do}\ C'_2, t |\!\rangle$, and $(C^*, \mathsf{while}\ B\ \mathsf{do}\ C'_2)$ in R.

(b) $C^* = \langle C; \mathsf{while}\ B\ \mathsf{do}\ C_2 \rangle V$ for some $C \in Com$ and possibly empty $V \in \boldsymbol{Com}$. From the operational semantics, we obtain $\langle\!| C_1, s |\!\rangle \twoheadrightarrow \langle\!| \langle C \rangle V, t |\!\rangle$. Since $C_1 \doteq C'_1$, there are $C' \in Com$ and $V' \in \boldsymbol{Com}$ with $\langle\!| C'_1, s |\!\rangle \twoheadrightarrow^* \langle\!| \langle C' \rangle V', t |\!\rangle$, $C \doteq C'$ and $V \doteq V'$. From the operational semantics, we obtain $\langle\!| C'_1; \mathsf{while}\ B\ \mathsf{do}\ C'_2, s |\!\rangle \twoheadrightarrow^* \langle\!| \langle C'; \mathsf{while}\ B\ \mathsf{do}\ C'_2 \rangle V', t |\!\rangle$ with $(C; \mathsf{while}\ B\ \mathsf{do}\ C_2\ ,\ C'; \mathsf{while}\ B\ \mathsf{do}\ C'_2) \in R$ (follows from $C \doteq C'$, $\mathsf{while}\ B\ \mathsf{do}\ C_2 \doteq \mathsf{while}\ B\ \mathsf{do}\ C'_2$, and the definition of R), and $V \doteq V'$.

Hence, R is a pointwise weak bisimulation up to $\doteq$.

4. Define R as the symmetric relation $\{(\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2\ ,\ \mathsf{if}\ B\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2) \mid C_1 \doteq C'_1, C_2 \doteq C'_2\}$.
Let $(\mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2\ ,\ \mathsf{if}\ B\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2) \in R$ and $s \in S$ be arbitrary. We make a case distinction on the value of B in s:

   (a) $\langle\!| B, s |\!\rangle \downarrow \mathsf{False}$: From the operational semantics $\langle\!| \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s |\!\rangle \twoheadrightarrow \langle\!| C_2, s |\!\rangle$ and $\langle\!| \mathsf{if}\ B\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2, s |\!\rangle \twoheadrightarrow \langle\!| C'_2, s |\!\rangle$ follow. By definition of R, we have $C_2 \doteq C'_2$.

   (b) $\langle\!| B, s |\!\rangle \downarrow \mathsf{True}$: From the operational semantics $\langle\!| \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2, s |\!\rangle \twoheadrightarrow \langle\!| C_1, s |\!\rangle$ and $\langle\!| \mathsf{if}\ B\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2, s |\!\rangle \twoheadrightarrow \langle\!| C'_1, s |\!\rangle$ follow. By definition of R, we have $C_1 \doteq C'_1$.

   Hence, R is a pointwise weak bisimulation up to $\doteq$.

5. The assertion follows directly from Definition 8.

Proof (of Theorem 2). We prove the first assertion for pointwise weak bisimulations (rather than weak bisimulations) by induction on the term structure of vectors of length 1, i.e. $V \in Com_{\mathcal{V}}$. By Lemma 5(5), the assertion is then lifted to arbitrary vectors in $\boldsymbol{Com_{\mathcal{V}}}$, and with Lemma 4, we obtain what we wanted. Let σ, ρ be substitutions that are preserving and ground for V.

1. Suppose V is skip or an assignment. Then $\sigma(V) = \rho(V) = V$, and the assertion follows by reflexivity of $\doteq$.
2. Suppose V is of the form $\alpha; C'$ or $C'; \alpha$. By induction hypothesis we have $\sigma(C') \doteq \rho(C')$. σ and ρ are preserving and ground for α, so we see $\sigma(\alpha) \doteq \rho(\alpha)$. From Lemma 5, $\sigma V \doteq \rho V$ follows.
3. Suppose $V = C_1; C_2$ with $C_1, C_2 \in Com_{\mathcal{V}}$. By induction hypothesis we have $\sigma C_1 \doteq \rho C_1$ and $\sigma C_2 \doteq \rho C_2$. From Lemma 5, $\sigma V \doteq \rho V$ follows

4. Suppose $V = \mathsf{while}\ B\ \mathsf{do}\ C'$ with $C' \in Com_{\mathcal{V}}$. By induction hypothesis we have $\sigma C' \doteq \rho C'$. From Lemma 5, $\sigma V \doteq \rho V$ follows.
5. Suppose $V = \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2$ with $C_1, C_2 \in Com_{\mathcal{V}}$. By induction hypothesis we have $\sigma C_1 \doteq \rho C_1$ and $\sigma C_2 \doteq \rho C_2$. From Lemma 5, $\sigma V \doteq \rho V$ follows.
6. Suppose $V = \mathsf{fork}(C_0\langle C_1, \ldots, C_n\rangle)$ with $C_i \in Com_{\mathcal{V}}$ for $i = 0, \ldots, n$. By induction hypothesis we have $\sigma C_i \doteq \rho C_i$ for $i = 0, \ldots, n$. From Lemma 5, we first obtain $\sigma\langle C_1, \ldots, C_n\rangle \doteq \rho\langle C_1, \ldots, C_n\rangle$ and then $\sigma V \doteq \rho V$.

The second assertion follows from part 1 and the observation that $\pi V' = V$ for every lifting V' of $V \in$ MWL.

B.3 Proof of Theorem 3

We first state and prove two lemmas to simplify reasoning with $\simeq_L$ on $Com_{\mathcal{V}}$.

Lemma 6. *If $V_1 \simeq_L V_2$ holds for two programs $V_1, V_2 \in \boldsymbol{Com_{\mathcal{V}}}$ then $\sigma V_1 \simeq_L \sigma V_2$ holds for each substitution σ that is preserving (but not necessarily ground).*

Proof. Given an arbitrary substitution η that is preserving and ground for σV_1 and σV_2, we obtain $\eta(\sigma V_1) \simeq_L \eta(\sigma V_2)$ from $V_1 \simeq_L V_2$, Definition 4, and the fact that $\eta \circ \sigma$ is preserving and ground for V_1 and V_2. Since η was chosen arbitrarily, $\sigma V_1 \simeq_L \sigma V_2$ follows.

Lemma 7. *Let $V, V', V_0, V_0', \ldots, V_n, V_n' \in \boldsymbol{Com_{\mathcal{V}}}$ be programs that may contain meta-variables. If $V_i \simeq_L V_i'$ holds for each $i \in \{0, \ldots, n\}$ according to Definition 4 and $V \simeq_L V'$ can be syntactically derived from the assumptions $V_0 \simeq_L V_0'$, $\ldots$, $V_n \simeq_L V_n'$ with the rules in Figure 1 then $V \simeq_L V'$ holds according to Definition 4.*

Proof. We argue by induction on the size of $\mathcal{D}$, the derivation of $V \simeq_L V'$ from $V_0 \simeq_L V_0'$, $\ldots$, $V_n \simeq_L V_n'$.

Base case: If $\mathcal{D}$ consists of zero rule applications then $V \simeq_L V'$ equals one of the assumptions.

Induction assumption: The proposition holds for all derivations with less than n rule applications.

Step case: Assume $\mathcal{D}$ consists of n rule applications. We make a case distinction on the last rule applied in $\mathcal{D}$. Here, we consider only the case where [*SComp*] is the last rule applied. The cases for the other rules can be shown along the same lines.

Let $C_1, C_1', C_2, C_2' \in Com_{\mathcal{V}}$ be arbitrary with $C_1 \simeq_L C_1'$ and $C_2 \simeq_L C_2'$. Let σ be an arbitrary substitution that is preserving and ground for C_1, C_1', C_2, C_2'. From $C_1 \simeq_L C_1'$, $C_2 \simeq_L C_2'$, and Definition 4 we obtain $\sigma C_1 \simeq_L \sigma C_1'$ and $\sigma C_2 \simeq_L \sigma C_2'$. An application of [*SComp*] (for ground programs) yields $(\sigma C_1; \sigma C_2) \simeq_L (\sigma C_1'; \sigma C_2')$. Since $\sigma(C_1; C_2) = (\sigma C_1; \sigma C_2)$, $(\sigma C_1'; \sigma C_2') = \sigma(C_1'; C_2')$, and σ was chosen freely, we obtain $C_1; C_2 \simeq_L C_1'; C_2'$ from Definition 4.

Proof (of Theorem 3). We prove the three propositions in different order.

2. By induction on the height of the given derivation of $V \hookrightarrow V' : S$, one obtains $V' = \rho V$ for some preserving substitution ρ. The assertion follows by applying Theorem 2.1 to $\sigma(V)$ and $(\sigma\rho)(V)$ for an arbitrary σ that is preserving and ground for both V and V'.
3. Since $V' \simeq_L S$ implies $V' \cong_L S$ according to Theorem 1, it suffices to show that $V \hookrightarrow V' : S$ implies $V' \simeq_L S$. We prove this second proposition by induction on the minimal height of the given derivation $\mathcal{D}$ of $V \hookrightarrow V' : S$.
Base case: $\mathcal{D}$ consists of a single rule application. We perform a case distinction on this rule:
[*Var*] We have $V = V' = S = \alpha$ for some meta-variable $\alpha \in \mathcal{V}$. Let σ be an arbitrary substitution that is preserving and ground for α. As $\sigma\alpha$ is a command in $Stut_{\mathcal{V}}$ that is free of meta-variables (i.e. a sequential composition of skip statements), we obtain $\sigma\alpha \simeq_L \sigma\alpha$ from [*Skip*] and [*SeqComp*] in Figure 1. Hence, $\alpha \simeq_L \alpha$ holds.
[*Skp*] We have $V = V' = S =$ skip. From [*Skip*] in Figure 1, we obtain skip $\simeq_L$ skip.
[Ass_h] We have $V = V' = Id{:=}Exp$ and $S =$ skip with $Id : high$. From [SHA_2] in Figure 1, we obtain $Id{:=}Exp \simeq_L$ skip.
[Ass_l] We have $V = V' = S = Id{:=}Exp$ with $Id : low$ and $Exp : low$. From [*LA*] in Figure 1, we obtain $Id{:=}Exp \simeq_L Id{:=}Exp$.
Induction assumption: For any derivation $\mathcal{D}'$ of a judgment $W \hookrightarrow W' : S'$ with height less than the height of $\mathcal{D}$, $W' \simeq_L S'$ holds.
Step case: We make a case distinction on the rule applied at the root of $\mathcal{D}$.
[*Seq*] We have $V = C_1; C_2$, $V' = C_1'; C_2'$, and $S = S_1; S_2$ with $C_1 \hookrightarrow C_1' : S_1$ and $C_2 \hookrightarrow C_2' : S_2$. By induction assumption, $C_1' \simeq_L S_1$ and $C_2' \simeq_L S_2$ hold. An application of [*SComp*] in Figure 1 yields $C_1'; C_2' \simeq_L S_1; S_2$.
[*Par*] We have $V = \langle C_1, \ldots, C_n \rangle$, $V' = \langle C_1', \ldots, C_n' \rangle$, and $S = \langle S_1, \ldots, S_n \rangle$ with $C_i \hookrightarrow C_i' : S_i$ for all $i \in \{1, \ldots, n\}$. By induction assumption, $C_i' \simeq_L S_i$ holds for all $i \in \{1, \ldots, n\}$. Application of [*PComp*] in Figure 1 yields $\langle C_1', \ldots, C_n' \rangle \simeq_L \langle S_1, \ldots, S_n \rangle$.
[*Frk*] We have $V = \mathsf{fork}(C_1 V_2)$, $V' = \mathsf{fork}(C_1' V_2')$, and $S = \mathsf{fork}(S_1 S_2)$ with $C_1 \hookrightarrow C_1' : S_1$ and $V_2 \hookrightarrow V_2' : S_2$. By induction assumption, $C_1' \simeq_L S_1$ and $V_2' \simeq_L S_2$ hold. An application of [*Fork*] in Figure 1 yields $\mathsf{fork}(C_1' V_2') \simeq_L \mathsf{fork}(S_1 S_2)$.
[*Whl*] We have $V =$ while B do C_1, $V' =$ while B do C_1', and $S =$ while B do S_1 with $B : low$ and $C_1 \hookrightarrow C_1' : S_1$. By induction assumption, $C_1' \simeq_L S_1$ holds. An application of [*WL*] in Figure 1 yields while B do $C_1' \simeq_L$ while B do S_1.
[$Cond_l$] We have $V =$ if B then C_1 else C_2, $V' =$ if B then C_1' else C_2', and $S =$ if B then S_1 else S_2 with $B : low$, $C_1 \hookrightarrow C_1' : S_1$, and $C_2 \hookrightarrow C_2' : S_2$. By induction assumption, $C_1' \simeq_L S_1$ and $C_2' \simeq_L S_2$ hold. An application of [*LIte*] in Figure 1 yields if B then C_1' else $C_2' \simeq_L$ if B' then S_1 else S_2.
[$Cond_h$] We have $V =$ if B then C_1 else C_2, $V' =$ if B then $\sigma C_1'$ else $\sigma C_2'$, and $S =$ skip$;\sigma S_1$ with $B : high$, $C_1 \hookrightarrow C_1' : S_1$, $C_2 \hookrightarrow C_2' : S_2$, and $\sigma \in \mathcal{U}(\{S_1 \simeq_L^? S_2\})$. By induction assumption, $C_1' \simeq_L S_1$ and $C_2' \simeq_L S_2$ hold.

From Lemma 6, we obtain $\sigma C_1' \doteq_L \sigma S_1$ and $\sigma C_2' \doteq_L \sigma S_2$ as σ is preserving. Then we obtain $\sigma C_2' \doteq_L \sigma S_1$ from $\sigma C_2' \doteq_L \sigma S_2$, $\sigma S_1 \doteq_L \sigma S_2$ (follows from $\sigma \in \mathcal{U}(\{S_1 \doteq^?_L S_2\})$), and the fact that $\doteq_L$ is symmetric and transitive. An application of $[SHIte_2]$ in Figure 1 yields $\mathsf{if}\ B\ \mathsf{then}\ \sigma C_1'\ \mathsf{else}\ \sigma C_2' \doteq_L \mathsf{skip}; \sigma S_1$.

1. From $V' \approxeq_L S$ and the symmetry of $\approxeq_L$, we obtain $S \approxeq_L V'$. Then $V' \approxeq_L V'$ follows from the transitivity of $\approxeq_L$.

B.4 Proof of Lemma 1

The language $Pad_{\mathcal{V}}$ is given by the following grammar:

$$P ::= X \mid \mathsf{skip} \mid Id_h{:=}Exp \mid P_1; P_2$$

Here X is a placeholder for meta-variables in $\mathcal{V}$, Id_h is a placeholder for program variables in *Var* with domain *high*, and P_1, P_2 are placeholders for commands in $Pad_{\mathcal{V}}$. We define $const(C) = |C|_{\mathsf{skip}} + \sum_{Id_h, Exp, Id_h : high} |C|_{Id_h := Exp}$, where $|C|_D$ denotes the number of occurrences of D as a subterm of C.

Proof (of Lemma 1). We show each direction of the implication:

($\Longrightarrow$) Assume $C_1 \doteq_L C_2$. Make a case distinction:

1. Assume $const(C_1) \neq const(C_2)$. Let σ be the substitution mapping all variables in C_1 and C_2 to ϵ. The judgment $\sigma C_1 \doteq_L \sigma C_2$ is not derivable with the rules in Figure 1. According to Definition 4, this contradicts the assumption $C_1 \doteq_L C_2$ as σ is preserving. Hence, this case is not possible.
2. Assume $const(C_1) = const(C_2)$ and $|C_1|_\alpha \neq |C_2|_\alpha$ for some meta-variable $\alpha \in \mathcal{V}$. Let σ be the substitution mapping α to skip and all other variables in C_1 and C_2 to ϵ. We have $const(\sigma C_1) \neq const(\sigma C_2)$ and, thus, a contradiction to the assumption $C_1 \doteq_L C_2$ (argue like in case 1).
3. Assume $const(C_1) = const(C_2)$ and $|C_1|_\alpha = |C_2|_\alpha$ for all meta-variables $\alpha \in \mathcal{V}$.

The above case distinction is complete as it covers all possible cases. Under the assumption $C_1 \doteq_L C_2$, only case 3. is possible. Hence, the implication holds.

($\Longleftarrow$) The argument is by induction on the number of meta-variables occurring in (C_1, C_2). Assume $const(C_1) = const(C_2)$ and $\forall \alpha \in \mathcal{V}$: $|C_1|_\alpha = |C_2|_\alpha$.

Base case: No meta-variables occur in (C_1, C_2). We obtain $C_1 \doteq_L C_2$ from $const(C_1) = const(C_2)$ and $C_1, C_2 \in Pad$ (argue by induction on $const(C_1)$).

Step case: $n+1$ meta-variables occur in (C_1, C_2) and the proposition holds for all command pairs with n or less meta-variables. Let α' be an arbitrary variable occurring in C_1 (and, hence, also in C_2). Let σ be an arbitrary substitution that is preserving and ground for C_1 and C_2. We decompose σ into two substitutions σ_1, σ_2 such that $\sigma = \sigma_2 \circ \sigma_1$, $dom\, \sigma_1 = \{\alpha'\}$, and $dom\, \sigma_2 = dom\, \sigma \setminus \{\alpha'\}$. Note that σ_1 and σ_2 both are preserving. We have $|\sigma_1(C_1)|_\alpha = |C_1|_\alpha = |C_2|_\alpha = |\sigma_1(C_2)|_\alpha$ for all $\alpha \neq \alpha'$. Moreover, we have $const(\sigma_1(C_1)) = const(\sigma_1(C_2))$ because $const(C_1) = const(C_2)$ and $|C_1|_{\alpha'} = |C_2|_{\alpha'}$. From the induction assumption (n meta-variables occur in $\sigma(C_1), \sigma(C_2)$), we obtain $\sigma_1(C_1) \doteq_L \sigma_1(C_2)$. This means, in particular, $\sigma_2(\sigma_1(C_1)) \doteq_L \sigma_2(\sigma_1(C_2))$ holds. Since $\sigma = \sigma_2 \circ \sigma_1$ and σ was chosen freely, we obtain $C_1 \doteq_L C_2$.

B.5 Proof of Theorem 4

Proof (Sketch). The proof of assertion 1 is a straightforward inductive argument over the structure of the derivation of $V \rightharpoonup \overline{V}$ and an inspection of each rule in Figure 3.

To simplify the argument in the proof of part 2, we introduce the auxiliary language $Mgl_{\mathcal{V}}$, defined by

$$L ::= P^t \mid P^t; Id_l{:=}Exp; L \mid P^t; \mathsf{if}\ B\ \mathsf{then}\ L_1\ \mathsf{else}\ L_2; L \\ \mid P^t; \mathsf{while}\ B\ \mathsf{do}\ L_1; L \mid P^t; \mathsf{fork}(L_1 V); L$$

where Id_l is a placeholder for program variables in Var with domain low, L, L_1, L_2 are placeholders for commands in $Mgl_{\mathcal{V}}$, V is a placeholder for a command vector in $\boldsymbol{Mgl_{\mathcal{V}}}$, and P^t is a placeholder for a command of the form X or of the form $P; X$ with $P \in Pad_{\mathcal{V}}$.

The theorem follows immediately from:

(a) $\mathcal{U}(\{V_1' \simeq_L^? V_2'\}) \neq \emptyset$ implies $\mathcal{U}(\{V_1^* \simeq_L^? V_2^*\}) \neq \emptyset$ for all liftings $V_1^*, V_2^* \in \boldsymbol{Mgl_{\mathcal{V}}}$ of V_1, V_2 such that (V_1^*, V_2^*) does not share meta-variables with (V_1', V_2') and each meta-variable occurs at most once in (V_1^*, V_2^*).

(b) $\mathcal{U}(\{V_1' \simeq_L^? V_1'\}) \neq \emptyset$ implies $\mathcal{U}(\{V_1^* \simeq_L^? V_1^*\}) \neq \emptyset$ for all liftings $V_1^* \in \boldsymbol{Mgl_{\mathcal{V}}}$ of V_1 such that V_1^* does not share meta-variables with V_1' and each meta-variable occurs at most once in V_1^*.

(c) $\overline{V_1}$ and $\overline{V_2}$ are in the language $\boldsymbol{Mgl_{\mathcal{V}}}$ and each meta-variable occurs at most once in $(\overline{V_1}, \overline{V_2})$.

These propositions are implied by Lemmas 8, 9, and 10, respectively, that we present in the following.

Lemma 8. *Let $V_i \in \boldsymbol{Com}$, with liftings $V_i' \in \boldsymbol{Com_{\mathcal{V}}}$ and $V_i^* \in \boldsymbol{Mgl_{\mathcal{V}}}$ for $i = 1, 2$. Furthermore, assume every meta-variable occurs at most once in (V_1^*, V_2^*) and that V_1^* and V_2^* do not share meta-variables with V_1' and V_2'. Then we have*

$$\mathcal{U}(\{V_1' \simeq_L^? V_2'\}) \neq \emptyset \Rightarrow \mathcal{U}(\{V_1^* \simeq_L^? V_2^*\}) \neq \emptyset$$

More precisely, we can find $\rho \in \mathcal{U}(\{V_1^* \simeq_L^? V_2^*\})$ with $dom(\rho) \subseteq var(V_1^*) \cup var(V_2^*)$.

Proof. Suppose σ is a preserving substitution with $\sigma V_1' \simeq_L \sigma V_2'$. We will inductively construct preserving substitutions ρ_1 with $\rho_1 V_1^* \simeq_L \sigma V_1'$, and ρ_2 with $\rho_2 V_2^* \simeq_L \sigma V_2'$ with the property $dom(\rho_i) \subseteq var(V_i^*)$ for $i = 1, 2$. The meta-variables in V_1^* and V_2^* are disjoint, so $\rho = \rho_1 \cup \rho_2$ is well–defined and a unifier of $V_1^* \simeq_L^? V_2^*$ because of $\rho V_1^* \simeq_L \sigma V_1' \simeq_L \sigma V_2' \simeq_L \rho V_2^*$. We prove the assertion by induction on the term structure of $V_1^* \in \boldsymbol{Mgl_{\mathcal{V}}}$, starting with $V_1^* = C_1^* \in Mgl_{\mathcal{V}}$ and hence $V_1 = C_1 \in Com$ and $V_1' = C_1' \in Com_{\mathcal{V}}$.

Suppose $C_1^* \in Pad_{\mathcal{V}} \cap Mgl_{\mathcal{V}}$. Then by definition of $Mgl_{\mathcal{V}}$, C_1^* contains at least one meta-variable α. C_1' is also a lifting of C_1, so it must be in $Pad_{\mathcal{V}} \cup \{\epsilon\}$. Let $\alpha_1, \ldots, \alpha_n$ be the meta-variables in C_1'. Define $\rho(\alpha) := \sigma(\alpha_1); \ldots; \sigma(\alpha_n)$, and set $\rho(Y) = \epsilon$ for all $Y \neq \alpha$ occuring in C_1^*. C_1^* and C_1' are both liftings of C_1, so they contain the same number of skips and assignments to high variables. By definition of ρ we see that $\sigma C_1'$ and ρC_1^* contain the same meta-variables and the same number of constants. Using the calculus in Figure 1 in combination with Lemma 1 we can conclude that $\rho C_1^* \simeq_L \sigma C_1'$. Furthermore, $dom(\rho) \subseteq var(C_1^*)$ is satisfied.

Suppose $C_1^* = P; \mathsf{if}\ B\ \mathsf{then}\ C_{1,1}^*\ \mathsf{else}\ C_{1,2}^*; C^*$. The command C_1' is also a lifting of C_1, so it has the structure $P'; \mathsf{if}\ B\ \mathsf{then}\ C_{1,1}'\ \mathsf{else}\ C_{1,2}'; C'$, with (possibly empty) commands P', C'.

If B is a *low* conditional, we inductively construct substitutions $\rho_1, \rho_2, \rho_3, \rho_4$ such that $\rho_1 P \simeq_L \sigma P'$, $\rho_2 C_{1,1}^* \simeq_L \sigma C_{1,1}'$, $\rho_3 C_{1,2}^* \simeq_L \sigma C_{1,2}'$ and $\rho_4 C^* \simeq_L \sigma C'$. The domains of the ρ_i are disjoint by the hypothesis that $dom(\rho_i)$ is a subset of the meta-variables of the corresponding subcommand and the assumption that every meta-variable occurs only once in C_1^*, so $\rho = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$ is well-defined. Using Lemma 7 we can conclude $\rho C_1^* = \rho_1 P; \mathsf{if}\ B\ \mathsf{then}\ \rho_2 C_{1,1}^*\ \mathsf{else}\ \rho_3 C_{1,2}^*; \rho_4 C^* \simeq_L \sigma P'; \mathsf{if}\ B\ \mathsf{then}\ \sigma C_{1,1}'\ \mathsf{else}\ \sigma C_{1,2}'; \sigma C' \simeq_L \sigma C_1'$. Furthermore, $dom(\rho) \subseteq var(C_1^*)$ is satisfied.

If B is a *high* conditional, the precondition $\sigma C_1' \simeq_L \sigma C_2'$ together with the definition of $\simeq_L$ on high conditionals shows that $\sigma C_{1,1}' \simeq_L \sigma C_{1,2}'$ holds. Applying induction hypothesis we obtain $\rho_{2,1}$ and $\rho_{2,2}$ with $\rho_{2,1} C_{1,1}^* \simeq_L \sigma C_{1,1}' \simeq_L \sigma C_{1,2}' \simeq_L \rho_{2,2} C_{1,2}^*$ and ρ_1, ρ_3 with $\rho_1 P \simeq_L \sigma P'$ and $\rho_3 C^* \simeq_L \sigma C'$. With $\rho = \rho_1 \cup \rho_{2,1} \cup \rho_{2,2} \cup \rho_3$ we have $\rho C_1^* = \rho_1 P; \mathsf{if}\ B\ \mathsf{then}\ \rho_{2,1} C_{1,1}^*\ \mathsf{else}\ \rho_{2,2} C_{1,2}^*; \rho_3 C^* \simeq_L \rho_1 P; \mathsf{skip}; \rho_{2,1} C_{1,1}^*; \rho_3 C^* \simeq_L \sigma P'; \mathsf{skip}; \sigma C_{1,1}'; \sigma C' \simeq_L \sigma P'; \mathsf{if}\ B\ \mathsf{then}\ \sigma C_{1,1}'\ \mathsf{else}\ \sigma C_{1,2}'; \sigma C' \simeq_L \sigma C_1'$. Furthermore, $dom(\rho) \subseteq var(C_1^*)$ is satisfied.

The remaining induction steps for $Mgl_{\mathcal{V}}$ and the lifting to $\boldsymbol{Mgl_{\mathcal{V}}}$ can be treated in the same way as the low conditional case.

Lemma 9. *Let $V \in \boldsymbol{Com}$, with liftings $V' \in \boldsymbol{Com_{\mathcal{V}}}$ and $V^* \in \boldsymbol{Mgl_{\mathcal{V}}}$. Furthermore, assume every meta-variable occurs at most once in V^* and that V^* does not share meta-variables with V'. Then we have*

$$\mathcal{U}(\{V' \simeq_L^? V'\}) \neq \emptyset \Rightarrow \mathcal{U}(\{V^* \simeq_L^? V^*\}) \neq \emptyset.$$

Proof. We will inductively construct ρ with $\rho V^* \simeq_L^? \rho V^*$ and $dom(\rho) \subseteq var(V^*)$, starting with $V^* = C^* \in Mgl_{\mathcal{V}}$ and $V' = C' \in Com_{\mathcal{V}}$

Suppose $C^* \in Pad_{\mathcal{V}}$. Then the identity is in $\mathcal{U}(\{C^* \simeq_L^? C^*\})$.

Suppose $C^* = P; l{:=}Exp; C_1^*$. C' is also a lifting of C, so $C' = P'; l{:=}Exp; C_1'$ with possibly empty P', C'. From $\sigma C' \simeq_L \sigma C'$, and the definition of $\simeq_L$ we know that $Exp : low$ and $\mathcal{U}(\{C_1' \simeq_L^? C_1'\}) \neq \emptyset$. We apply induction hypothesis to obtain $\rho \in \mathcal{U}(\{C_1^* \simeq_L^? C_1^*)$. By Lemma 7 we obtain $\rho C^* \simeq_L \rho C^*$.

Suppose $C^* = P; \mathsf{if}\ B\ \mathsf{then}\ C_1^*\ \mathsf{else}\ C_2^*; C_3^*$ with $B : low$. C' is also a lifting of C, so $C' = P'; \mathsf{if}\ B\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'; C_3'$. From $\sigma C' \simeq_L \sigma C'$, and the definition of $\simeq_L$ we know that $\mathcal{U}(\{C_i' \simeq_L^? C_i'\}) \neq \emptyset$ for $i = 1, 2, 3$. We apply induction hypothesis to obtain $\rho_i \in \mathcal{U}(\{C_i^* \simeq_L^? C_i^*\})$ for $i = 1, 2, 3$. $\rho = \rho_1 \cup \rho_2 \cup \rho_3$ is well–defined as the domains are pairwise disjoint, and $\rho \in \mathcal{U}(\{C^* \simeq_L^? C^*\})$.

Suppose $C^* = P;$ if B then C_1^* else $C_2^*; C_3^*$ with $B : high$. We then know that $C' = P';$ if B then C_1' else $C_2'; C_3'$ because both C' and C^* are liftings of C. From $\sigma C' \simeq_L \sigma C'$, and the definition of $\simeq_L$ we have $\sigma C_1' \simeq_L \sigma C_2'$. By Lemma 8 we obtain ρ_1 with $\rho_1 C_1^* \simeq_L \rho_1 C_2^*$. (Note that every meta-variable occurs at most once in (C_1^*, C_2^*).) Applying induction hypothesis to P and C_3^* we obtain ρ_0 with $\rho_0 P \simeq_L \rho_0 P$ and ρ_2 with $\rho_2 C_3^* \simeq_L \rho_2 C_3^*$. With $\rho = \rho_0 \cup \rho_1 \cup \rho_2$ we have $\rho C^* \simeq_L \rho C^*$, which is what we wanted. The remaining induction steps for $Mgl_\mathcal{V}$ and the lifting to $\boldsymbol{Mgl_\mathcal{V}}$ can be treated in the same way as the low conditional case.

Lemma 10. *Let $V \in \boldsymbol{Com}$ and $\overline{V} \in \boldsymbol{Com_\mathcal{V}}$. If we have $V \rightharpoonup \overline{V}$, then $\overline{V} \in \boldsymbol{Mgl_\mathcal{V}}$ and every meta-variable occurs at most once in $\overline{V}$.*

Proof. By induction on the term structure of $V \in \boldsymbol{Com}$: First, suppose $V = C \in Com$. If $C =$ skip, we have $C \rightharpoonup$ skip$;X$, which is in $Mgl_\mathcal{V}$. The same holds for $C = h{:=}Exp$. If $C = l{:=}Exp$, we have $C \rightharpoonup X; l{:=}Exp; Y$, which is in $Mgl_\mathcal{V}$. If $C =$ while B do C', we have $C \rightharpoonup \overline{C} = X;$ while B do $\overline{C'}; Y$ with $C' \rightharpoonup \overline{C'}$. By induction hypothesis, $\overline{C'} \in Mgl_\mathcal{V}$, and by definition of $Mgl_\mathcal{V}$ we have $\overline{C} \in Mgl_\mathcal{V}$. If $C =$ fork$(C'V)$, we have $C \rightharpoonup \overline{C} = X;$ fork$(\overline{C'V}); Y$ with $C' \rightharpoonup \overline{C'}$ and $V \rightharpoonup \overline{V}$. By induction hypothesis, $\overline{C'} \in Mgl_\mathcal{V}$, and $\overline{V} \in \boldsymbol{Mgl_\mathcal{V}}$ and by definition of $Mgl_\mathcal{V}$ we have $\overline{C} \in Mgl_\mathcal{V}$. If $C =$ if B then C_1 else C_2, we have $C \rightharpoonup \overline{C} = X;$ if B then $\overline{C_1}$ else $\overline{C_2}; Y$ with $C_1 \rightharpoonup \overline{C_1}$ and $C_2 \rightharpoonup \overline{C_2}$. By induction hypothesis, $\overline{C_1}, \overline{C_2} \in Mgl_\mathcal{V}$, and by definition of $Mgl_\mathcal{V}$ we have $\overline{C} \in Mgl_\mathcal{V}$. Suppose now $C = C_1; C_2$. Let $C_1 \rightharpoonup \overline{C_1}$, and let $C_2 \rightharpoonup \overline{C_2}$. By induction hypothesis $\overline{C_1}, \overline{C_2} \in Mgl_\mathcal{V}$. As $\overline{C_1} \in Mgl_\mathcal{V}$, we can write it as (implicit induction on $Mgl_\mathcal{V}$) $\overline{C_1} = \overline{C_1'}; P; X$ for a maximal $P \in Pad_\mathcal{V}$ and $X \in \mathcal{V}$. Observe that $P; \overline{C_2} \in Mgl_\mathcal{V}$, and thus $\overline{C_1'}; P; \overline{C_2} \in Mgl_\mathcal{V}$. This is what we wanted, as we have $C_1; C_2 \rightharpoonup \overline{C_1'}; P; \overline{C_2}$.

If $V = \langle C_1, \ldots, C_n \rangle \in \boldsymbol{Com}$, we have $V \rightharpoonup \overline{V} = \langle \overline{C_1}, \ldots, \overline{C_n} \rangle$ with $C_i \rightharpoonup \overline{C_i}$ for $i = 1, \ldots, n$. By induction hypothesis, $\overline{C_1}, \ldots, \overline{C_n} \in Mgl_\mathcal{V}$, and thus $\overline{V} \in \boldsymbol{Mgl_\mathcal{V}}$. The assertion that each meta-variable occurs at most once follows from the requirement that each meta-variable inserted while lifting must be *fresh*.

B.6 Proof of Lemma 2

Proof. We prove a stronger assertion. In addition to the statement of Lemma 2 we prove that $dom(\eta) \cup var(ran(\eta)) \subseteq var(V_1) \cup var(V_2)$, and that $dom(\eta) \cap var(ran(\eta)) = \emptyset$ hold, where $var(V)$ denotes the set of all meta-variables occurring in V. We proceed by structural induction on the derivation tree $\mathcal{D}$ of the judgment $V_1 \simeq^?_L V_2 :: \eta$.

If $\mathcal{D}$ consists of an application of the rule $[Var_1]$ we have $V_1 = \alpha$ and $V_2 = C$. The assertion follows, as α does not occur in $var(C)$ by assumption. If $\mathcal{D}$ consists of an application of the rule $[Asg]$ the assertion follows directly by definition of $\simeq_L$. If the root of $\mathcal{D}$ is an application of rule $[Seq_1]$, we have $V_1 = \alpha; C_1$ and $V_2 = C_2$, and $C_1 \simeq^?_L C_2 :: \eta$. By hypothesis, $\eta C_1 \simeq_L \eta C_2$ holds. $\eta[\alpha \backslash \epsilon]\alpha; C_1 = \eta C_1 \simeq_L \eta C_2 \simeq_L \eta[\alpha \backslash \epsilon] C_2$, as α does not occur in C_1, C_2. $[Seq_1']$ follows similarly. If the root of $\mathcal{D}$ is an application of rule $[Seq_2]$, we have $V_1 = \mathsf{skip}; C_1$ and $V_2 = \mathsf{skip}; C_2$, and $C_1 \simeq^?_L C_2 :: \eta$. By hypothesis, $\eta C_1 \simeq_L \eta C_2$ holds. Then, by Lemma 7 we also have $\eta(\mathsf{skip}; C_1) \simeq_L \eta(\mathsf{skip}; C_2)$. If the root of $\mathcal{D}$ is the application of $[Ite]$, we have $V_1 = \mathsf{if}\ B_1\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2$ and $V_2 = \mathsf{if}\ B_2\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2$, together with $B_1 \equiv B_2$ and $C_1 \simeq^?_L C'_1 :: \eta_1$, $C_2 \simeq^?_L C'_2 :: \eta_2$. By hypothesis we have $\eta_i \in \mathcal{U}(\{C_i \simeq^?_L C'_i\})$, and $dom(\eta_i) \cup var(ran(\eta_i)) \subseteq var(C_i) \cup var(C'_i)$ for $i = 1, 2$. As $(var(C_1) \cup var(C'_1))$ and $(var(C_2) \cup var(C'_2))$ are disjoint by hypothesis, $\eta = \eta_1 \cup \eta_2$ is well-defined and $dom(\eta) \cup var(ran(\eta)) \subseteq var(V_1) \cup var(V_2)$ and $dom(\eta) \cap var(ran(\eta)) = \emptyset$ hold. With the help of Lemma 7 we see that η is indeed a unifier.

The remaining cases can be proved along the same lines.

B.7 Proof of Theorem 5

The proof of Theorem 5 proceeds via the following landmarks, where the language $Mgl_{\mathcal{V}}$ is defined like in Appendix B.5.

(a) If $C_1, C_2 \in Mgl_{\mathcal{V}}$, if every meta-variable occurs at most once in (C_1, C_2) and if $\mathcal{U}(\{C_1 \simeq^?_L C_2\}) \neq \emptyset$ and $C_i \hookrightarrow' C'_i : S_i$ for $i = 1, 2$, then $S_1, S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$, $S_1 \doteq S_2$ and every meta-variable occurs at most once in (S_1, S_2).

(b) If $S_1, S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and $S_1 \doteq S_2$ then there is an η with $S_1 \simeq^?_L S_2 :: \eta$.

We first show in full detail how Theorem 5 can be inferred from the above propositions. Then we give the definition of the relation $\doteq$ and justify the propositions with Lemmas 12 and 11.

Proof (of Theorem 5:).

1. Let W be an arbitrary lifting of V. From Theorem 4 follows that $\sigma W \simeq_L \sigma W$ implies $\mathcal{U}(\overline{V} \simeq_L \overline{V}) \neq \emptyset$. From Lemma 10 we see that $\overline{V} \in Mgl_{\mathcal{V}}$ and that every meta-variable occurs at most once in $\overline{V}$.
   Restricting ourselves to commands in $Com_{\mathcal{V}}$ for the moment and substituting C for $\overline{V}$, it suffices to show the assertion
   $$\exists\sigma.\sigma C \simeq_L \sigma C \Rightarrow C \hookrightarrow' C' : S$$
   for all $C \in Mgl_{\mathcal{V}}$ where every meta-variable occurs at most once. We proceed by induction on the term structure of C.
   If $C \in Pad_{\mathcal{V}}$, we always have $C \hookrightarrow' C' : S$.
   If $C = P; Id_l{:=}Exp; C_1$, and $\sigma C \simeq_L \sigma C$, then we have $\sigma P \simeq_L \sigma P$ and $\sigma C_1 \simeq_L \sigma C_1$ and $Exp : low$ by definition of $\simeq_L$. By applying induction hypothesis we obtain $P \hookrightarrow' P' : S_0$ and $C_1 \hookrightarrow' C'_1 : S_1$. By definition of $\hookrightarrow'$ this implies $C \hookrightarrow' P'; Id_l{:=}Exp; C'_1 : S_0; Id_l{:=}Exp; S_1$.
   If $C = P; \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2; C_3$ with $B : low$, and $\sigma C \simeq_L \sigma C$, then we have $\sigma P \simeq_L \sigma P$ and $\sigma C_i \simeq_L \sigma C_i$ for $i = 1, 2, 3$. By applying induction hypothesis we obtain $P \hookrightarrow' P' : S_0$ and $C_i \hookrightarrow' C'_i : S_i$ for $i = 1, 2, 3$. By definition of $\hookrightarrow'$ this implies $C \hookrightarrow' P'; \mathsf{if}\ B\ \mathsf{then}\ C'_1\ \mathsf{else}\ C'_2; C'_3 : S_0; \mathsf{if}\ B\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2; S_3$

$$\frac{P, P' \in Stut_{\mathcal{V}} \cup \{\epsilon\}}{P \doteq P'} \qquad \frac{P, P' \in Stut_{\mathcal{V}} \cup \{\epsilon\} \quad C \doteq C' \quad Id : low \quad Exp_1 \equiv Exp_2}{P; Id{:=}Exp_1; C \doteq P'; Id{:=}Exp_2; C'}$$

$$\frac{P, P' \in Stut_{\mathcal{V}} \cup \{\epsilon\} \quad C_1 \doteq C_1' \quad C_2 \doteq C_2' \quad B_1 \equiv B_2}{P; \mathsf{while}\ B_1\ \mathsf{do}\ C_1; C_2 \doteq P'; \mathsf{while}\ B_2\ \mathsf{do}\ C_1'; C_2'} \qquad \frac{P, P' \in Stut_{\mathcal{V}} \cup \{\epsilon\} \quad C_1 \doteq C_1' \quad C_2 \doteq C_2' \quad V \doteq V'}{P; \mathsf{fork}(C_1 V); C_2 \doteq P'; \mathsf{fork}(C_1' V'); C_2'}$$

$$\frac{P, P' \in Stut_{\mathcal{V}} \cup \{\epsilon\} \quad C_i \doteq C_i' \quad i = 1,2,3 \quad B_1 \equiv B_2}{P; \mathsf{if}\ B_1\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2; C_3 \doteq P'; \mathsf{if}\ B_2\ \mathsf{then}\ C_1'\ \mathsf{else}\ C_2'; C_3'} \qquad \frac{C_1 \doteq C_1', \ldots, C_n \doteq C_n'}{\langle C_1, \ldots, C_n \rangle \doteq \langle C_1', \ldots, C_n' \rangle}$$

Fig. 7. Resembling Commands

If $C = P; \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2; C_3$ with $B : high$, and $\sigma C \simeq_L \sigma C$, then we have $\sigma P \simeq_L \sigma P$ and $\sigma C_i \simeq_L \sigma C_i$ for $i = 1, 3$. Furthermore we have $\sigma C_1 \simeq_L \sigma C_2$, from which we get $\sigma C_2 \simeq_L \sigma C_2$ by transitivity and symmetry of $\simeq_L$. Induction hypothesis yields $P \hookrightarrow' P' : S_0$ and $C_i \hookrightarrow' C_i' : S_i$ for $i = 1, 2, 3$. Every meta-variable occurs at most once in C, hence the same holds true for the subterms C_1, C_2.
Proposition **(a)** shows that $S_1, S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and $S_1 \doteq S_2$ and that every meta-variable occurs at most once in (S_1, S_2). Proposition **(b)** implies that there is η with $S_1 \simeq^?_L S_2 :: \eta$. η is a unifier of S_1, S_2, as Lemma 2 shows. We conclude $C \hookrightarrow' P'; \mathsf{if}\ B\ \mathsf{then}\ \eta C_1'\ \mathsf{else}\ \eta C_2'; C_3' : S_0; \mathsf{skip}; \eta S_1; S_3$, which is what we wanted.
The cases for the other constructors follow along the same lines as the low conditional. The assertion can then simply be lifted to command vectors.

2. By a straightforward induction over the derivation tree it follows that $W \hookrightarrow W' : S$ implies $W' = \sigma W$ for a preserving substitution σ. In the proof of Theorem 3 it was shown that $S \simeq_L W'$ holds. By symmetry and transitivity of $\simeq_L$ we obtain $\sigma W \simeq_L \sigma W$. The assertion now follows directly from part 1. of Theorem 5.

To simplify the atomic treatment of subcommands in $Stut_{\mathcal{V}}$ of commands in $Slice_{\mathcal{V}}$ in inductive arguments, we introduce the language $Slice^+_{\mathcal{V}}$. (Note the resemblance to the definition of $Mgl_{\mathcal{V}}$). Define the set $Slice^+_{\mathcal{V}}$ by the grammar:

$$\begin{aligned} L ::= {} & P \mid P; Id_l{:=}Exp; L \mid P; \mathsf{if}\ B\ \mathsf{then}\ L_1\ \mathsf{else}\ L_2; L \\ & \mid P; \mathsf{while}\ B\ \mathsf{do}\ L_1; L \mid P; \mathsf{fork}(L_1 V); L \end{aligned}$$

where L, L_1, L_2 are placeholders for commands in $Slice^+_{\mathcal{V}}$, V is a placeholder for a command vector in $\boldsymbol{Slice^+_{\mathcal{V}}}$, and P is a placeholder for a command in $Stut_{\mathcal{V}} \cup \{\epsilon\}$. By a straightforward induction one proves that $Slice_{\mathcal{V}} \subseteq Slice^+_{\mathcal{V}}$.

Definition 10. *The binary relation $\doteq$ on $\boldsymbol{Slice^+_{\mathcal{V}}}$ is defined as the reflexive, symmetric and transitive closure of the relation inductively defined in Figure 7. We call commands $V, V' \in \boldsymbol{Mgl_{\mathcal{V}}}$ with $V \doteq V'$* resembling.

Lemma 11. *If $S_1, S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and $S_1 \doteq S_2$ then there is an η with $S_1 \simeq^?_L S_2 :: \eta$.*

Proof. By induction on the term structure of S_1:

Suppose $S_1 \in Stut_{\mathcal{V}}$. Then by definition of $\doteq$, S_2 must also be in $Stut_{\mathcal{V}}$. $S_1, S_2 \in Mgl_{\mathcal{V}}$, hence they have terminal meta-variables. A simple induction on the length of S_1 shows that $S_1 \simeq_L S_2 :: \eta$ is derivable.

Suppose $S_1 = P_1; \textsf{if } B_1 \textsf{ then } S_{1,1} \textsf{ else } S_{1,2}; S_{1,3}$, where $P_1, S_{1,1}, S_{1,2}, S_{1,3}$ are elements of $Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$. By definition of $\doteq$ and $Mgl_{\mathcal{V}}$ we know that $S_2 = P_2; \textsf{if } B_2 \textsf{ then } S_{2,1} \textsf{ else } S_{2,2}; S_{2,3}$ with $P_2, S_{2,1}, S_{2,2}, S_{2,3} \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and $P_1 \doteq P_2$, $S_{1,i} \doteq S_{2,i}$ for $i = 1, 2, 3$, and $B_1 \equiv B_2$. We apply induction hypothesis and obtain $P_1 \simeq^?_L P_2 :: \sigma_0$, and $S_{1,i} \simeq^?_L S_{2,i} :: \sigma_i$ for $i = 1, 2, 3$. We can conclude $S_1 \simeq^?_L S_2 :: \sigma$ with $\sigma = \bigcup_{i=0}^{3} \sigma_i$ by definition of the unification calculus.

The other cases follow along the same lines.

Lemma 12. *1. If $C \in Mgl_{\mathcal{V}}$ with $C \hookrightarrow' C' : S$ and if every meta-variable occurs at most once in C, then $S \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and $var(S) \subseteq var(C)$ and every meta-variable occurs at most once in S.*

2. If $C_1, C_2 \in Mgl_{\mathcal{V}}$ with $\mathcal{U}(\{C_1 \simeq^?_L C_2\}) \neq \emptyset$ and $C_i \hookrightarrow' C'_i : S_i$ for $i = 1, 2$, then $S_1 \doteq S_2$.

Proof. 1. It is easy to see that $S \in Slice_{\mathcal{V}}$ holds, so we concentrate on containment in $Mgl_{\mathcal{V}}$. We proceed by induction on the term structure of $C \in Mgl_{\mathcal{V}}$. Suppose $C \in Pad_{\mathcal{V}} \cap Mgl_{\mathcal{V}}$. By definition of $Mgl_{\mathcal{V}}$, C has a terminal meta-variable. Then clearly $S \in Mgl_{\mathcal{V}}$ as it contains a terminal meta-variable and no assignments. The condition on the meta-variables is fulfilled.

Suppose now $C = P; \textsf{if } B \textsf{ then } C_1 \textsf{ else } C_2; C_3$ with $B : low$ and $C \hookrightarrow' C' : S$. Then by definition of $\hookrightarrow'$ we have $P \hookrightarrow' P' : S_0$, $C_1 \hookrightarrow' C'_1 : S_1$, $C_2 \hookrightarrow' C'_2 : S_2$ and $C_3 \hookrightarrow' C'_3 : S_3$. By induction hypothesis, $S_0, S_1, S_2, S_3 \in Mgl_{\mathcal{V}}$ and the condition on the meta-variables holds. By definition of $Mgl_{\mathcal{V}}$ $S = S_0; \textsf{if } B \textsf{ then } S_1 \textsf{ else } S_2; S_3 \in Mgl_{\mathcal{V}}$, and the condition on the meta-variables follows.

Suppose now $C = P; \textsf{if } B \textsf{ then } C_1 \textsf{ else } C_2; C_3$ with $B : high$. We have $C \hookrightarrow' C' : S$, so by definition of $\hookrightarrow'$ we have $P \hookrightarrow' P' : S_0$, $C_1 \hookrightarrow' C'_1 : S_1$, $C_2 \hookrightarrow' C'_2 : S_2$ and $C_3 \hookrightarrow' C'_3 : S_3$ and also $S_1 \simeq_L S_2 :: \eta$ for some η. By induction hypothesis, $S_1, S_2, S_3 \in Mgl_{\mathcal{V}}$ and the condition on the meta-variables holds. As $var(S_i) \subseteq var(C_i)$, (S_1, S_2) contains every meta-variable at most once. With Lemma 13 we see that $\eta S_1 \in Mgl_{\mathcal{V}}$, and every meta-variable occurs at most once in ηS_1. From the proof of Lemma 2 it follows that $dom(\eta) \cup ran(var(\eta))$ is a subset of the meta-variables in S_1 and S_2 and hence the condition on the meta variables is fulfilled for $S = (S_0; \textsf{skip}; \eta S_1); S_3$. $S_0; \textsf{skip} \in Stut_{\mathcal{V}}$, and so by the definition of $Mgl_{\mathcal{V}}$, $S_0; \textsf{skip}; \eta S_1 \in Mgl_{\mathcal{V}}$. By a straightforward induction one shows that $D_1; D_2 \in Mgl_{\mathcal{V}}$ whenever $D_1, D_2 \in Mgl_{\mathcal{V}}$, and so we conclude that $S = (S_0; \textsf{skip}; \eta S_1); S_3 \in Mgl_{\mathcal{V}}$.

The cases for the other constructors follow along the same lines as the low conditional.

2. Therefore let $\sigma C_1 \simeq_L \sigma C_2$. By symmetry and transitivity of $\simeq_L$ we conclude $\sigma C_1 \simeq_L \sigma C_1$ and $\sigma C_2 \simeq_L \sigma C_2$. With help of Lemma 15 we obtain $C''_1, C''_2 \in$

$Slice_{\mathcal{V}}$ with $S_1 \doteq C_1'' \simeq_L \sigma C_1 \simeq_L \sigma C_2 \simeq_L C_2'' \doteq S_2$. Lemma 14 shows that $C_1'' \simeq_L C_2''$ implies $C_1'' \doteq C_2''$, and by transitivity of $\doteq$ we get $S_1 \doteq S_2$.

Lemma 13. *If $S_1, S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ with $S_1 \simeq_L S_2 :: \eta$ and if every variable occurs at most once in (S_1, S_2) then $\eta S_1, \eta S_2 \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and every meta-variable occurs at most once in ηS_1.*

Proof. We prove the assertion by induction on the term structure of S_1.

Suppose $S_1 \in Stut_{\mathcal{V}}$. Only the rules [*Var₁*],[*Var₂*],[*Seq₁*], [*Seq′₁*] and [*Seq₂*] apply, so $S_2 \in Stut_{\mathcal{V}}$. From $S_1, S_2 \in Mgl_{\mathcal{V}}$ we see that both commands contain a terminal variable. The two base cases for the derivation, [*Var₁*] and [*Var₂*] map the terminal variable at the end of one command to the end of the other command. With this definition of η we see that both ηS_1 and ηS_2 have terminal variables and hence are elements of $Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$. As (S_1, S_2) contains every meta-variable only once, the same holds for ηS_1.

Suppose $S_1 = P_1; \mathsf{if}\ B_1\ \mathsf{then}\ S_{1,1}\ \mathsf{else}\ S_{1,2}; S_{1,3}$ with $B : low$. From $S_1 \simeq_L^? S_2 :: \eta$ and definition of the rules [*Ite*] [*Seq₃*] and [*Seq₄*] in Figure 4 it follows that $S_2 = P_2; \mathsf{if}\ B_2\ \mathsf{then}\ S_{2,1}\ \mathsf{else}\ S_{2,2}; S_{2,3}$ with $B_1 \equiv B_2$ and that $P_1 \simeq_L^? P_2 :: \eta_0$, $S_{1,1} \simeq_L^? S_{2,1} :: \eta_1$, $S_{1,2} \simeq_L^? S_{2,2} :: \eta_2$, and $S_{1,3} \simeq_L^? S_{2,3} :: \eta_3$ are derivable. By induction hypothesis have $\eta_0 P_1$, $\eta_1 S_{1,1}$, $\eta_2 S_{1,2}$, $\eta_3 S_{1,3} \in Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$. With $\eta = \eta_0 \cup \eta_1 \cup \eta_2 \cup \eta_3$ and the fact that the domains and variable ranges of the η_i are mutually disjoint (see the proof of Lemma 2) we have $\eta S_1 = \eta_0 P_1; \mathsf{if}\ B_1\ \mathsf{then}\ \eta_1 S_{1,1}\ \mathsf{else}\ \eta_2 S_{1,2}; \eta_3 S_{1,3}$, which is in $Mgl_{\mathcal{V}} \cap Slice_{\mathcal{V}}$ and contains every meta-variable at most once.

All other constructors can be treated in the same way as the low conditional.

Lemma 14. *Let $S_1, S_2 \in Slice_{\mathcal{V}}$. Then we have*

$$S_1 \simeq_L S_2 \Rightarrow S_1 \doteq S_2.$$

Proof. We proceed by induction on the structure of S_1, where we make use of the fact that $Slice_{\mathcal{V}} \subseteq Slice^+_{\mathcal{V}}$. If $S_1 \in Stut_{\mathcal{V}}$, then by definition of $\simeq_L$ and the precondition $S_2 \in Slice_{\mathcal{V}}$ we know that $S_2 \in Stut_{\mathcal{V}}$, and hence $S_1 \doteq S_2$. If $S_1 = P_1; \mathsf{if}\ B_1\ \mathsf{then}\ S_{1,1}\ \mathsf{else}\ S_{1,2}; S_{1,3}$ with $B_1 : low$, then by definition of $\simeq_L$ we know that $S_2 = P_2; \mathsf{if}\ B_2\ \mathsf{then}\ S_{2,1}\ \mathsf{else}\ S_{2,2}; S_{2,3}$ with $P_1, P_2 = \epsilon$ or $P_1 \simeq_L P_2$, and $S_{1,i} \simeq_L S_{2,i}$ for $i = 1, 2, 3$ and $B_1 \equiv B_2$. By definition of $\doteq$, $P_1 \doteq P_2$ holds, and by induction hypothesis we see $S_{1,i} \doteq S_{2,i}$ for $i = 1, 2, 3$. By definition of $\doteq$ we conclude $S_1 \doteq S_2$. The other constructors follow in a similar fashion.

Lemma 15. *Let $C \in Mgl_{\mathcal{V}}$ with $\sigma \in \mathcal{U}(\{C \simeq_L C\})$ and $C \hookrightarrow' C' : S$. Then there is $C'' \in Slice_{\mathcal{V}}$ with $C'' \simeq_L \sigma C$ and $C'' \doteq S$.*

Proof. We proceed by structural induction on $C \in Mgl_{\mathcal{V}}$.

Suppose $C \in Pad_{\mathcal{V}}$. Choose C'' as σC, where all assignments to high variables are replaced by skips. We have $C'' \in Stut_{\mathcal{V}}$. We also have $S \in Stut_{\mathcal{V}}$, and so $C'' \doteq S$.

Suppose $C = P; \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2; C_3$ with $B : low$. We have $\sigma C \simeq_L \sigma C$, so by definition of $\simeq_L$ we obtain $\sigma P \simeq_L \sigma P$ and $\sigma C_i \simeq_L \sigma C_i$ for $i = 1, 2, 3$.

From the precondition $C \hookrightarrow' C' : S$ and the definition of $\hookrightarrow'$ we get $P \hookrightarrow' P' : S_0$ and $C_i \hookrightarrow' C_i' : S_i$ for $i = 1, 2, 3$. We apply induction hypothesis to the corresponding command-pairs and obtain P'', C_i'' with $P'' \simeq_L \sigma P$, $P'' \doteq S_0$, and $C_i'' \simeq_L \sigma C_i$, $C_i'' \doteq S_i$ for $i = 1, 2, 3$. Then we can conclude $C'' = P''; \mathsf{if}\ B\ \mathsf{then}\ C_1''\ \mathsf{else}\ C_2''; C_3'' \simeq_L \sigma C$, as well as $C'' \doteq S$.

Suppose $C = P; \mathsf{if}\ B\ \mathsf{then}\ C_1\ \mathsf{else}\ C_2; C_3$ with $B : \mathit{high}$. We have $\sigma C \simeq_L \sigma C$, so by definition of $\simeq_L$ we obtain $\sigma P \simeq_L \sigma P$, $\sigma C_1 \simeq_L \sigma C_1$ $\sigma C_3 \simeq_L \sigma C_3$ and $\sigma C_1 \simeq_L \sigma C_2$. By symmetry and transitivity of $\simeq_L$ we conclude $\sigma C_2 \simeq_L \sigma C_2$. From the precondition $C \hookrightarrow' C' : S$ and the definition of $\hookrightarrow'$ we get $P \hookrightarrow' P' : S_0$ and $C_i \hookrightarrow' C_i' : S_i$ for $i = 1, 2, 3$. We apply induction hypothesis to the corresponding command-pairs and obtain P'', C_i'' with the desired properties for $i = 1, 2, 3$. Define C'' as $P''; \mathsf{skip}; C_1''; C_3'' \in \mathit{Slice}_{\mathcal{V}}$. We have $C'' \simeq_L \sigma C$ by Lemma 7. On the other hand, $S = S_0; \mathsf{skip}; \eta S_1; S_3$ for some preserving substitution η. We are left to show $S \doteq C''$. From hypothesis we know $C_1'' \doteq S_1$ and $C_3'' \doteq S_3$. A straightforward induction shows that $S_1 \doteq \eta S_1$ for every preserving η. Another straightforward induction shows that $\doteq$ is a congruence with respect to sequential composition and so we conclude $C'' = P''; \mathsf{skip}; C_1''; C_3'' \doteq S_0; \mathsf{skip}; \eta S_1; S_3 = S$.

The other constructors follow along the same lines as the low conditional.